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Gradient Einstein solitons

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ABSTRACT

In this paper we consider a perturbation of the Ricci solitons equation proposed by J. P. Bourguignon. We show that these structures are more rigid than standard Ricci solitons. It turns out that this property holds also in the Lorentzian setting and for a more general class of structures which includes some gravitational theories. We prove several classification results both in the compact and the noncompact case and we provide at the same time existence results for rotationally symmetric solutions.

1. Introduction and statement of the results

One of the most significant functional in Riemannian geometry is the Einstein–Hilbert action

$$g \longmapsto \mathcal{E}(g) = \int_M R\, dV_g,$$

where M^n, $n \geq 3$, is a n-dimensional compact differentiable manifold, g is a Riemannian metric on M^n and R is its scalar curvature. It is well known that critical points of this functional on the space of metrics with fixed volume are Einstein metrics (see [1, Chapter 4]). In principle, it would be natural to use the associated (unnormalized) gradient flow

$$\partial_t g = -2\Big(Ric - \tfrac{1}{2} R\, g\Big) \tag{1.1}$$

to search for critical metrics. On the other hand, it turns out that such a flow is not parabolic. Hence, a general existence theory, even for short times, is not guaranteed by the present literature. This was one of

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the main reasons which led Hamilton to introduce the Ricci flow $\partial_t g = -2Ric$ in [23]. The Ricci flow has been studied intensively in recent years and plays a key role in Perelman's proof of the Poincaré conjecture (see [30–32]). For an introduction to Ricci flow, we refer the reader to [19].

An important aspect in the treatment of the Ricci flow is the study of Ricci solitons, which generate self-similar solutions to the flow and often arise as singularity models. Gradient Ricci solitons are Riemannian manifolds satisfying

$$Ric + \nabla^2 f = \lambda\, g,$$

for some smooth function f and some constant $\lambda \in \mathbb{R}$. For a complete survey on this subject, which has been treated by many authors, we refer the interested reader to [6,7].

Motivated by the notion of Ricci solitons, it is natural to consider special solutions to the flow (1.1), whose existence can be proved by ad hoc arguments. In particular, in this paper, we introduce the notion of *gradient Einstein solitons*. These are Riemannian manifolds satisfying

$$Ric - \frac{1}{2} R\, g + \nabla^2 f = \lambda\, g,$$

for some smooth function f and some constant $\lambda \in \mathbb{R}$. As expected, Einstein solitons as well generate self-similar solutions to the Einstein flow (1.1).

More in general, it is natural to consider on a Riemannian manifold (M^n, g), $n \geq 3$, geometric flows of the type

$$\partial_t g = -2(Ric - \rho R\, g), \tag{1.2}$$

for some $\rho \in \mathbb{R}$, $\rho \neq 0$. In [13] we developed the parabolic theory for these flows, which was first considered by Bourguignon in [3]. We call these flows Ricci–Bourguignon flows. Here we just notice that we can prove short time existence for every $-\infty < \rho < 1/2(n-1)$. However, as far as the subject of our investigation are self-similar solutions, every value of ρ is allowed. Associated to these flows, we have the following notion of gradient ρ-Einstein solitons.

Definition 1.1. Let (M^n, g), $n \geq 3$, be a Riemannian manifold and let $\rho \in \mathbb{R}$, $\rho \neq 0$. We say that (M^n, g) is a *gradient ρ-Einstein soliton* if there exists a smooth function $f : M^n \to \mathbb{R}$, such that the metric g satisfies the equation

$$Ric + \nabla^2 f = \rho R\, g + \lambda g, \tag{1.3}$$

for some constant $\lambda \in \mathbb{R}$.

We say that the soliton is *trivial* whenever ∇f is parallel. As usual, the ρ-Einstein soliton is steady for $\lambda = 0$, shrinking for $\lambda > 0$ and expanding for $\lambda < 0$. The function f is called a *ρ-Einstein potential* of the gradient ρ-Einstein soliton.

Corresponding to special values of the parameter ρ, we refer to the ρ-Einstein solitons with different names, according to the Riemannian tensor which rules the flow. Hence, for $\rho = 1/2$ we will have *gradient Einstein soliton*, for $\rho = 1/n$ *gradient traceless Ricci soliton* and for $\rho = 1/2(n-1)$ *gradient Schouten soliton*. In the compact case, arguments based on the maximum principle yield the following triviality result (listed below as Corollary 3.2), for solitons corresponding to these special values of ρ.

Theorem 1.1. *Every compact gradient Einstein, Schouten or traceless Ricci soliton is trivial.*

To deal with the noncompact case, it is useful to introduce the following notion of *rectifiability*. We say that a smooth function $f : M^n \to \mathbb{R}$ is *rectifiable* in an open set $U \subset M^n$ if and only if $|\nabla f_{|U}|$ is constant along every regular connected component of the level sets of $f_{|U}$. In particular, it can be seen that $f_{|U}$ only