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Asymptotic decay of solutions to 3D MHD equations

Yinxia Wang*

School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China

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ABSTRACT

In the very recent work Wang and Wang (2014) proved that the initial value problem for the three dimension incompressible MHD equations has a global solutions $(u, B) \in C([0,\infty); \chi^{-1})$, provided that the norms of the initial data are bounded exactly by the minimal value of the viscosity coefficients. In this paper, we prove that global solutions u, B that was obtained in Wang and Wang (2014) is asymptotic to zero in sense of the norm of the space χ^{-1} as time goes to infinity. Moreover, stability of global solutions is also discussed.

1. Introduction

This paper investigates the viscous incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + u\cdot\nabla u - B\cdot\nabla B - \mu\Delta u + \nabla\left(p + \frac{1}{2}|B|^2\right) = 0, \\ \partial_t B + u\cdot\nabla B - B\cdot\nabla u - \nu\Delta B = 0, \\ \nabla\cdot u = 0, \qquad \nabla\cdot B = 0 \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \ u = u^0(x), \qquad B = B^0(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

Here $u = (u^1, u^2, u^3), B = (B^1, B^2, B^3)$ and $P = p + \frac{1}{2}|B|^2$ denote the flow velocity, the magnetic field and the total kinetic pressure, respectively. $\mu > 0$ is the viscosity and $\nu > 0$ is the magnetic diffusivity.

The 3D MHD equations govern the dynamics of electrically conducting fluids. Examples of such fluids include plasmas, liquid metals, and salt water. MHD theory has the broad applications to the many branches

* Tel.: +86 37169310033.
E-mail address: wangyinxia@ncwu.edu.cn.

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of the sciences, e.g., geophysics, astrophysics, and engineering problems (see [15]). The local well-posedness of the MHD equations (1.1), (1.2) was established by Sermange and Temam [22], where the authors also proved the global well-posedness in 2D case. But whether this unique local solution to 3D MHD equations can exist globally is a challenge open problem in the mathematical fluid mechanics. Lei [12] proved the global existence of axially symmetric solutions to the systems of incompressible ideal magnetohydrodynamics and resistive magnetohydro-dynamics in three dimensions in the case that the magnetic fields are purely swirling and perpendicular to the velocity field. The authors [9] proved it would keep small it the difference between the magnetic field and the velocity is small initially, which results in a global strong solution without the smallness restriction on the size of initial velocity or magnetic field. Global existence of solutions to the problem (1.1), (1.2) with small initial data have been established, we refer to [17,16,24,23,25]. We notice that spaces where global solutions are constructed for small initial value, are necessarily scaling invariant spaces. Global well-posedness for the problem (1.1), (1.2) in BMO^{-1} for small data as well as local well-posedness in bmo^{-1} for small data were established in [17]. Later, Miao and Yuan [16] proved global well-posedness for the problem (1.1), (1.2) in homogeneous Besov space $\dot{B}_{p,r}^{-1+\frac{n}{p}}(1 \leq p < \infty, 1 \leq r \leq \infty)$ for small initial data. Wang and Wang [24] considered three dimensional incompressible MHD equations with mixed partial dissipation and magnetic diffusion. They proved that if $\|u_0\|_{H^1} + \|b_0\|_{H^1}$ is suitably small, then the three dimensional MHD equations with mixed partial dissipation and magnetic diffusion admit global smooth solutions. It should be noted, in all these works, that the norms in corresponding spaces of the initial data are assumed to be very small, smaller than the viscosity coefficient multiplied by a tiny positive constant ε. In the very recent work [25], Wang and Wang established global existence of solutions for the problem (1.1), (1.2) in the space χ^{-1} (see Section 2), provided that the norms of the initial data in χ^{-1} are bounded exactly by the minimal value of the viscosity coefficients, which extends the Navier–Stokes's result [13] to the problem (1.1), (1.2). The main result in [25] is the following global existence result.

Theorem 1.1 (*[25]*)**.** *Assume that* $u^0, B^0 \in \chi^{-1}$ *satisfy*

$$\|u^0\|_{\chi^{-1}} + \|B^0\|_{\chi^{-1}} < \min\{\mu, \nu\}.$$

The initial value problem (1.1), (1.2) *has a unique global solution* $(u, B) \in C((0, +\infty); \chi^{-1}) \bigcap L^1((0, +\infty); \chi^1)$. *Moreover,*

$$\sup_{0 \leq t < \infty} \left(\|u(t)\|_{\chi^{-1}} + \|B(t)\|_{\chi^{-1}}\right) + \left(\mu - \|u^0\|_{\chi^{-1}} - \|B^0\|_{\chi^{-1}}\right) \int_0^t \|u(t)\|_{\chi^1} dt$$
$$+ \left(\nu - \|u^0\|_{\chi^{-1}} - \|B^0\|_{\chi^{-1}}\right) \int_0^t \|B(t)\|_{\chi^1} dt \leq \|u^0\|_{\chi^{-1}} + \|B^0\|_{\chi^{-1}}. \tag{1.3}$$

For asymptotic behavior of solutions to the MHD equations, Schonbek et al. [20] proved that any weak solution (u, B) to (1.1), (1.2) in $\mathbb{R}^n (n \geq 2)$ satisfying either $\hat{u}_0(0) \neq 0$ or $\hat{B}_0(0) \neq 0$ decays in the L^2-norm as $(1+t)^{-n/2}$; when the initial data (u_0, B_0) satisfy $\hat{u}_0(0) = 0$ and $\hat{B}_0(0) = 0$, the solution decays in the L^2-norm as $(1+t)^{-n/2-1}$. Long time behavior of solutions to the MHD equations without magnetic diffusion was also discussed in Agapito and Schonbek [1]. But asymptotic behavior of global solutions in χ^{-1} that has been established in [25] is still open. Inspired by Benameur [3], we investigate asymptotic behavior and stability of global solutions in χ^{-1} in the current paper that has been established by Wang and Wang [25]. Our results extend the Navier–Stokes equations' results in [3] to MHD equations. For asymptotic behavior of solutions to the Navier–Stokes equations, lots of interesting results were established, we may refer to [2,18,19,21,31]. Our first result implies that global solution is asymptotic to zero in sense of the norm of the space χ^{-1} when $t \to \infty$. We state our first main results as follows.