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Existence of solutions for a higher order Kirchhoff type problem with exponential critical growth

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ABSTRACT

A higher order Kirchhoff type equation

$$\int_{\mathbb{R}^{2m}}\left(|\nabla^m u|^2+\sum_{\gamma=0}^{m-1}a_\gamma(x)\,|\nabla^\gamma u|^2\right)dx \times\left((-\Delta)^m u+\sum_{\gamma=0}^{m-1}(-1)^\gamma\nabla^\gamma\cdot(a_\gamma(x)\nabla^\gamma u)\right) =\frac{f(x,u)}{|x|^\beta}+\epsilon h(x)\quad \text{in } \mathbb{R}^{2m}$$

is considered in this paper. We assume that the nonlinearity of the equation has exponential critical growth and prove that, for a positive ϵ which is small enough, there are two distinct nontrivial solutions to the equation. When $\epsilon = 0$, we also prove that the equation has a nontrivial mountain-pass type solution.

1. Introduction and main results

Let $\nabla^\gamma u,\ \gamma\in\{0,1,2,\dots,m\}$, be the γ-th order gradients of a function $u\in W^{m,2}(\mathbb{R}^{2m})$ which are defined by

$$\nabla^\gamma u := \begin{cases}\Delta^{\frac{\gamma}{2}}u & \gamma \text{ even},\\ \nabla\Delta^{\frac{\gamma-1}{2}}u & \gamma \text{ odd}.\end{cases}$$

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Here and throughout this paper, $m \geq 2$ is an even integer and we use the notations that

$$\Delta^0 u = \nabla^0 u = u.$$

Consider the following nonlinear functional

$$J_\epsilon(u) = \frac{1}{4}\left(\int_{\mathbb{R}^{2m}}\left(|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x)|\nabla^\gamma u|^2\right)dx\right)^2 - \int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}dx - \epsilon\int_{\mathbb{R}^{2m}} hudx \tag{1.1}$$

which is related to a higher order nonlocal partial differential equation

$$\int_{\mathbb{R}^{2m}}\left(|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x)|\nabla^\gamma u|^2\right)dx\left((-\Delta)^m u + \sum_{\gamma=0}^{m-1}(-1)^\gamma \nabla^\gamma\cdot(a_\gamma(x)\nabla^\gamma u)\right) = \frac{f(x,u)}{|x|^\beta} + \epsilon h. \tag{1.2}$$

Here ϵ is a nonnegative constant, $h(x) \not\equiv 0$ belongs to the dual space of E which will be defined later, $0 \leq \beta < 2m$ and $a_\gamma(x)$ are continuous functions satisfying

(A_1) there exist positive constants a_γ, $\gamma = 0, 1, 2, \ldots, m-1$, such that $a_\gamma(x) \geq a_\gamma$ for all $x \in \mathbb{R}^{2m}$;
(A_2) $(a_0(x))^{-1} \in L^1(\mathbb{R}^{2m})$.

Since the equation contains an integral over $\mathbb{R}^{2m}$, it is no longer a pointwise identity and should be dealt with as a nonlocal problem. We call (1.2) a higher order Kirchhoff type equation because it is related to the stationary analog of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where ρ, ρ_0, h, E and L are constants. This equation was presented by Kirchhoff [19] as an extension of the classical D'Alembert wave equation for free vibrations of elastic string produced by transverse vibrations. This kind of nonlocal problem also appears in other fields, for example, biological systems where u describes a process which depends on the average of itself (for instance, population density). One can refer to [3,4,29] and the references therein for more details. After the work of Lions [26], where a functional analysis approach was proposed to this kind of equations, various models of Kirchhoff type have been studied by many authors using the variational framework, see, for example, [5,7–11,15,16,18,21–25,27,35,36], and the references therein. In particular, Li and Yang [23] studied the following equation and proved the existence of at least two positive solutions.

$$\begin{cases} M\left(\int_{\mathbb{R}^N}(|\nabla u|^N + V(x)|u|^N)dx\right)(-\Delta_N u + V(x)|u|^{N-2}u) = \lambda A(x)|u|^{p-2}u + f(u) \quad x \in \mathbb{R}^N, \\ u \in W^{1,N}(\mathbb{R}^N), \end{cases}$$

where $\Delta_N u = div(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator of u, $M(s) = s^k$ for $k > 0$, $s \geq 0$, $1 < p < N$, $\lambda > 0$ is a real parameter, $A(x)$ is a positive function in $L^\sigma(\mathbb{R}^N)$ with $\sigma = \frac{N}{N-p}$, V is a potential function and f is a nonlinearity term having critical exponential growth.

On the other hand, similar variational methods are also used to study equations without the nonlocal integral. For example, see [1,2,6,12–14,17,20,28,31–34,37–39] and the references therein. Among these results, the first author and Chang [39] proved an Adams type inequality and applied it to get the multiplicity result of a higher order quasilinear equation as follows

$$(-\Delta)^m u + \sum_{\gamma=0}^{m-1}(-1)^\gamma\nabla^\gamma\cdot(a_\gamma(x)\nabla^\gamma u) = \frac{f(x,u)}{|x|^\beta} + \epsilon h(x). \tag{1.4}$$