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Global well-posedness for the nonlinear wave equation with a cubic nonlinearity in two space dimensions

Wei Han

Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, PR China

A R T I C L E I N F O

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A B S T R A C T

In this paper, we prove the global well-posedness for the nonlinear wave equation with a cubic nonlinearity $u_{tt} - \Delta u = -u^3$ in two space dimensions, with initial data $(u_0, u_1) \in \dot{H}^s_x \times \dot{H}^{s-1}_x$, $1 > s > \frac{5}{11} \simeq 0.4545$.

1. Introduction

In this paper, we shall study the initial value problem for the nonlinear wave equation with a cubic nonlinearity in two dimensions:

$$\begin{cases} u_{tt} - \Delta u = -u^3, & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ (u, u_t)|_{t=0} = (u_0, u_1) \in \dot{H}^s_x \times \dot{H}^{s-1}_x(\mathbb{R}^2), \end{cases} \tag{1.1}$$

$u(t,x)$ is a real-valued function on $I \times \mathbb{R}^2$ and $0 \in I \subset \mathbb{R}$ is a time interval. The nonlinear wave equation is an important second order partial differential equation for the description of waves, as they occur in physics, such as sound waves, light waves and water waves.

The problem (1.1) is $L^2 \times \dot{H}^{-1}$-critical. In the sense that if u is a solution of problem (1.1), then so is

$$u_\lambda(t,x) = \lambda u(\lambda t, \lambda x), \quad \lambda > 0.$$

Moreover, the map $u \mapsto u_\lambda$ maps a solution of (1.1) to another solution of (1.1) with the initial data

$$(u_{0,\lambda}(x), u_{1,\lambda}(x)) = \big(\lambda u_0(\lambda x), \lambda^2 u_1(\lambda x)\big),$$

E-mail address: sh_hanweiwei1@126.com.

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and

$$\|(u_\lambda, \partial_t u_\lambda)\|_{L_t^\infty\left(L_x^2\times \dot{H}_x^{-1}\right)} = \|(u, u_t)\|_{L_t^\infty\left(L_x^2\times \dot{H}_x^{-1}\right)}. \tag{1.2}$$

In the case $s = 1$, the solutions obey the conserved *energy*,

$$E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{2}|u_t(t)|^2 + \frac{1}{4}|u(t)|^4\right) dx = E(u_0, u_1).$$

The problem (1.1) has been shown by Lindblad and Sogge [7] to be local well-posedness in $\dot{H}^s$, $s \geq 1/4$. Actually, Lindblad and Sogge discussed the problem of well-posedness in $\dot{H}^s \times \dot{H}^{s-1}$ for the equation

$$u_{tt} - \Delta u = \lambda |u|^{p-1} u, \quad t \in R, x \in R^n,$$

and proved local well-posedness in $\dot{H}^s \times \dot{H}^{s-1}$ when $n = 2$ and $3 < p \leq 5$, where

$$s = \frac{n+1}{4} - \frac{1}{p-1}.$$

Note that this value of s is $1/4$ for $n = 2, p = 3$, and this case was handled by Nakamura and Ozawa [8]. Moreover, it follows from Theorem 6.1 of Jiang, Wang and Yu [5] that if (u_0, u_1) is radially symmetric about the origin, then the result remains true even for $s \in [1/6, 1/4)$. It seems very hard to go "below" $\dot{H}^{1/6} \times \dot{H}^{-5/6}$ (namely, $s < 1/6$) even for radially symmetric data. The current paper discusses whether these local solutions exist globally in time and therefore we focused on the case of the lower regularity $s < 1$, and solved the problem for $s \in (5/11, 1)$. (Naturally, for $s = 1$, we have global existence of solutions thanks to the conservation law of the energy.) The problem remains open for $s \in [1/4, 5/11]$. More precisely, for any $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$ for $s \geq 1/4$, there exist a time $T = T(\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}})$ and a unique solution-pair (u, u_t) of (1.1) in a certain Banach space $X \subset C([0, T); \dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2))$, moreover, the solution map is continuous from $\dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$ to $C([0, T); \dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2))$. In particular, if these hold for any time T, we call that the problem (1.1) is globally well-posed.

By local theory and the conservation of energy, the global well-posedness in the energy space $(\dot{H}^1 \cap L^4) \times L^2$ can be immediately obtained. But for the initial data in $\dot{H}^s \times \dot{H}^{s-1}$, $s < 1$ ($u_0 \in L^4$ is needed sometimes), there is no conservation to give the *a priori* estimates of $\dot{H}^s \times \dot{H}^{s-1}$. In this paper, we will use the I-method to study the global well-posedness in the space $\dot{H}^s \times \dot{H}^{s-1}$, $s < 1$.

For the cubic nonlinear wave equation problem (1.1) in three dimensional case, many researchers studied this problem, such as [4,6,9,11]. To our knowledge, the best global result is global well-posedness in $\dot{H}^s \times \dot{H}^{s-1}$ for $1 > s > 0.7$ proved by Wu [11].

In this paper, we use the I-method, which is proposed by I-team [2,1,3], and combine it with the argument in [9,11,12] where the main idea is so-called the "linear–nonlinear decomposition" developed in [9]. However, because of the weak dispersive properties, the two dimensional case is more difficult than the three dimensional case. In fact, in order to break through $s < \frac{1}{2}$ we need to consider more situations, and also more nonlinear analysis techniques will be used to treat these situations.

Our main result is the following:

Theorem 1.1. *Let $1 > s > \frac{5}{11}$, and let the initial data either $(u_0, u_1) \in (\dot{H}^s \cap L^4)(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$ for $s > \frac{1}{2}$; or $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$ for $\frac{5}{11} < s \leq \frac{1}{2}$. Then the problem (1.1) is globally well-posed in $\dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$ for $1 > s > \frac{5}{11}$.*

Remark 1.1. As in [11], in the case of $s > \frac{1}{2}$, the initial data should be in $L^4(\mathbb{R}^2)$, so that the modified energy in (1.6) is well-defined, however, when $s \leq \frac{1}{2}$, the limitation of $L^4(\mathbb{R}^2)$ is not necessary.