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Boundary layers for the 3D primitive equations in a cube: The supercritical modes

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ABSTRACT

In this article we study the boundary layers for the viscous Linearized Primitive Equations (LPEs) when the viscosity is small. The LPEs are considered here in a cube. Besides the usual boundary layers that we analyze here too, corner layers due to the interaction between the different boundary layers are also studied.

1. Introduction

Various local areas models (LAM) are used for short term (5–7 days) weather predictions. When the Primitive Equations (PEs) are used, it is customary to use the PEs without viscosity since, usually, the effects or viscosity are felt after a period of about one week. However, the use of the inviscid Primitive Equations is confronted to the challenging problem of finding suitable boundary conditions for these equations. The issue is explained, from the practitioner point of view in the tutorial article [48]. From the mathematical point of view the issue is to find suitable boundary conditions which make these equations well-posed. In particular it is known since the work of Oliger and Sundström [31] (see also [44]) that the inviscid Primitive Equations are not well-posed for any set of local boundary equations unlike the related Euler equations of inviscid flows.

Traditionally the issue of the boundary equations for the inviscid Primitive Equations in a cube, is solved, in the linear case by considering a vertical expansion of these equations, which we recall and use below. Then

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suitable boundary conditions can be found for each of the modes of this expansion. It is explained in [17] that each mode of this expansion is identical to an inviscid shallow water equation and [17] contains a systematic study of the boundary conditions which are suitable for the inviscid linear shallow water equations; this work is being extended to the nonlinear case in e.g. [15,16,18].

In a certain sense works like [17] give a sufficient answer to the LAM boundary condition problem, although a systematic implementation remains to be done. Assuming that the limited domain is embedded in a larger one which is computed with a coarser mesh, we know, which boundary data to extract from the coarse grid simulation to be used in the limited domain.

In relation with these practical issues we address in this article another delicate problem: namely to determine the boundary conditions for the inviscid Primitive Equations which result, from the viscous case, by letting the viscosity go to zero. This cannot be done directly, as far as we know, and the route that we follow here is to determine the different boundary layers generated, at small viscosity, by the viscous Primitive Equations and then we can obtain the desired convergences. More precisely, we surmise the boundary conditions for the inviscid PEs and then confirm this result by a convergence theorem.

Since the works of Lions, Temam and Wang on the local and global existence and regularity of solutions of the Primitive Equations (PEs) (see [26,27]), many advances have been done in the understanding of these equations which constitute a simplified model of the Navier–Stokes equations in the compressible case for the atmosphere, in the incompressible case for the ocean. Despite these simplifications, appearing after certain approximations as e.g. the Boussinesq or the hydrostatic approximation, the PEs continue to be the object of many theoretical and numerical mathematical studies. In this article, we continue the analysis of the boundary layers of the Linearized Primitive Equations (LPEs) as introduced in our previous works in dimension 2 in [10] and dimension 3 with periodicity in one or two directions in [12]. In the present work we deal with less regular domains. More precisely, we consider domains which contain corners restricting thus ourselves, in a first step, to the cube. Besides its concordance with many physical situations, the cube domain is numerically advantageous to handle computations for several applications like weather predictions among many others of course. Nevertheless, the particularity of the domain (here the cube) generates a new type of boundary layers called *corner layers* in addition to the boundary layers of classical types. The mathematical study of corner layers is very delicate in general; see e.g. the pioneering work [42]. Furthermore, these studies were mainly considered in the case of stationary linear problems. In order to understand the dynamics of the viscous PEs solution, when the viscosity goes to zero and up to the boundary (including the corners), we use the method of correctors that we construct in an explicit manner. Hence, the convergence results, valid up to the boundary, are then proven in suitable spaces that we will specify later on in this work.

We start the article by setting the viscous primitive equations and their expansion in the vertical direction mode by mode (see Section 2). Then, in Section 3, we formally derive the inviscid limit problem for the viscous mode solutions when the viscosity is set to be 0. In Section 4, we construct the correctors and in Section 5 we prove some estimates for these correctors. Finally, in Section 6, we state and prove the main result which shows the convergence between the viscous and inviscid LPEs solutions confirming thus the validity of our somehow heuristical choices for the correctors. This study concerns all the vertical modes $n > n_c$, i.e. the supercritical modes (see (2.10)). The vertical modes $0 \leq n \leq n_c$ which necessitate different methods will be studied in a separate work. The mode zero $n = 0$ of the LPEs, (2.11) and (3.2), seems similar to the linearized Euler equations but the solutions behave differently in some respect (see e.g. [3] for the inviscid LPEs, mode zero). The subcritical modes $1 \leq n \leq n_c$ of the LPEs, see (2.12) and (3.1), are quite similar to the supercritical modes. However, the locations of some boundary layers and some inflows are different. The additional difficulties related to the subcritical boundary layers will be explained in more detail in Remark 6.1.

Of course this work makes systematic use of a number of tools and developments in boundary layer theory and in the study of the inviscid Primitive Equations. Important works in the mathematical analysis of