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Renormalized solutions to nonlinear parabolic problems in generalized Musielak–Orlicz spaces

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A B S T R A C T

We will present the proof of existence of renormalized solutions to a nonlinear parabolic problem $\partial_t u - \operatorname{div} a(\cdot, Du) = f$ with right-hand side f and initial data u_0 in L^1. The growth and coercivity conditions on the monotone vector field a are prescribed by a generalized $\mathcal{N}$-function M which is anisotropic and inhomogeneous with respect to the space variable. In particular, M does not have to satisfy an upper growth bound described by a Δ_2-condition. Therefore we work with generalized Musielak–Orlicz spaces which are not necessarily reflexive. Moreover we provide a weak sequential stability result for a more general problem: $\partial_t\beta(\cdot, u) - \operatorname{div}(a(\cdot, Du) + F(u)) = f$, where β is a monotone function with respect to the second variable and F is locally Lipschitz continuous. Within the proof we use truncation methods, Young measure techniques, the integration by parts formula and monotonicity arguments which have been adapted to nonreflexive Musielak–Orlicz spaces.

1. Introduction

1.1. Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with Lipschitz boundary $\partial\Omega$ if $d \geq 2$ and let $[0, T]$ be a finite time interval, and $Q_T = (0, T) \times \Omega$. We are interested in existence of renormalized solutions to the following nonlinear parabolic problem

$$\begin{aligned} &\partial_t\beta(x, u(t, x)) - \operatorname{div}(a(x, Du(t, x)) + F(u(t, x))) = f \quad \text{in } Q_T, \\ &u = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ &\beta(x, u(0, x)) = b_0 \quad \text{in } \Omega, \end{aligned} \tag{P, f, b_0}$$

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where $f \in L^1(Q_T), F : \mathbb{R} \to \mathbb{R}^d$ is locally Lipschitz continuous and

B1: $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone (with respect to the second argument), single-valued Carathéodory function.
B2: $\beta(x, 0) = 0$ for a.a. $x \in \Omega$.
B3: for all $l \in \mathbb{R}$ $\beta(\cdot, l) \in L^1(\Omega)$.

Moreover, we assume that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following conditions:

A1: $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function.
A2: there exist generalized $\mathcal{N}$-functions $M, M^* : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, where M^* is a conjugate function to M (for definitions see Section 2), a constant $c_a \in (0, 1]$ and a nonnegative integrable function a_0 such that

$$a(x, \xi) \cdot \xi \geq c_a \{M^*(x, a(x, \xi)) + M(x, \xi)\} - a_0(x) \tag{1}$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.
A3: $a(\cdot, \cdot)$ is monotone, i.e.

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq 0 \tag{2}$$

for a.a. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^d$.

Additionally, we assume that

M1: there exist $c_M > 0, \nu > 0$ and $\xi_0 \in \mathbb{R}^d$ such that

$$M(x, \xi) \geq c_M |\xi|^{1+\nu} \quad \text{for a.a. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^d,\ |\xi| \geq |\xi_0|. \tag{3}$$

M2: the conjugate function

$$M^* \text{ satisfies the } \Delta_2\text{-condition}, \tag{4}$$

i.e., there exist some nonnegative, integrable on Ω function g_{M^*} and a constant $C_{M^*} > 0$ such that

$$M^*(x, 2\xi) \leq C_{M^*} M^*(x, \xi) + g_{M^*}(x) \quad \text{for all } \xi \in \mathbb{R}^d \text{ and a.a. } x \in \Omega. \tag{5}$$

M3: the conjugate function M^* satisfies

$$\lim_{|\xi| \to \infty} \operatorname*{ess\,inf}_{x \in \Omega} \frac{M^*(x, \xi)}{|\xi|} = \infty. \tag{6}$$

Example 1.1. The following examples of $\mathcal{N}$-functions fit into our setting:

- $M(x, \xi) = |\xi|^{p(x)}$, with $p : \Omega \to (p^-, \infty)$ measurable and $p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x) > 1$.
- $M(x, \xi) = \sum_{i=1}^d |\xi_i|^{p_i(x)}, p_i : \Omega \to (p_i^-, \infty)$ measurable, $p_i^- := \operatorname{ess\,inf}_{x \in \Omega} p_i(x) > 1, i = 1, \dots, d$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$.