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Modulational instability in the Whitham equation with surface tension and vorticity

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ABSTRACT

We study modulational stability and instability in the Whitham equation, combining the dispersion relation of water waves and a nonlinearity of the shallow water equations, and modified to permit the effects of surface tension and constant vorticity. When the surface tension coefficient is large, we show that a periodic traveling wave of sufficiently small amplitude is unstable to long wavelength perturbations if the wave number is greater than a critical value, and stable otherwise, similarly to the Benjamin–Feir instability of gravity waves. In the case of weak surface tension, we find intervals of stable and unstable wave numbers, whose boundaries are associated with the extremum of the group velocity, the resonance between the first and second harmonics, the resonance between long and short waves, and a resonance between dispersion and the nonlinearity. For each constant vorticity, we show that a periodic traveling wave of sufficiently small amplitude is unstable if the wave number is greater than a critical value, and stable otherwise. Moreover it can be made stable for a sufficiently large vorticity. The results agree with those based upon numerical computations or formal multiple-scale expansions to the physical problem.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$u_t + \sqrt{gd}\Big(1 + \frac{1}{6}d^2\partial_x^2\Big)u_x + uu_x = 0 \tag{1.1}$$

approximates the water wave problem in a small amplitude and long wavelength regime, and furthermore, it satisfactorily explains solitary waves and cnoidal wave trains of the physical problem. Here, $t \in \mathbb{R}$ is proportional to elapsed time and $x \in \mathbb{R}$ is the spatial variable in the predominant direction of wave propagation; $u = u(x,t)$ is real-valued, related to the surface displacement from the undisturbed fluid depth

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d, and g is the constant due to gravitational acceleration. Throughout we express partial differentiation either by a subscript or using the symbol ∂. When waves are long compared to the fluid depth so that $kd \ll 1$, k the wave number, one may expand the phase velocity[1] for the water wave problem in the irrotational setting and write that

$$c_{WW}(k) := \pm\sqrt{\frac{g\tanh(kd)}{k}} = \pm\sqrt{gd}\Big(1 - \frac{1}{6}k^2d^2\Big) + O(k^4d^4). \tag{1.2}$$

Therefore the KdV equation may be regarded as to approximate up to second order the dispersion relation of the water wave problem in the long wavelength regime. Here and elsewhere, $\pm$ mean left and right moving waves. (One may take, without loss of generality, the + sign in (1.1). As a matter of fact, $x \mapsto -x$ and $u \mapsto -u$ turn one equation to the other.)

However the KdV equation misses peaking into sharp crests. Furthermore waves in shallow water at times break into bores whereas the KdV equation prevents singularity formation from solutions. This is not surprising since the phase velocity associated with the linear part of the KdV equation poorly approximates[2] c_{WW} when kd becomes large. One may incidentally argue that peaking and breaking are high frequency phenomena, for which short wave components become important and the long wavelength assumption is no longer adequate.

Whitham therefore emphasized in [38] that "it is intriguing to know what kind of simpler mathematical equation (than the physical problem) could include" breaking and peaking, and he put forward

$$u_t + \mathcal{M}u_x + uu_x = 0, \tag{1.3}$$

where $\mathcal{M}$ is a Fourier multiplier, defined via its symbol as

$$\widehat{\mathcal{M}f}(k) = m(k)\widehat{f}(k), \tag{1.4}$$

and $m = c_{WW}$ (see (1.2)). It combines the full range of dispersion of water waves, rather than a second order approximation, and the nonlinearity of the shallow water equations, and hence it may offer an improvement over the KdV equation for short and intermediately long waves. As a matter of fact, numerical experiments in [5,31] indicate that Whitham's model approximates waves in water on par with or better than the KdV or other shallow water equations do in some respects outside the long wavelength regime. (The KdV equation, on the contrary, seems a better approximation in the long wavelength regime.) Moreover Whitham conjectured that (1.3)–(1.4), where $m = c_{WW}$, would explain breaking and peaking. Wave breaking – bounded solutions with unbounded derivatives – in the Whitham equation for water waves was recently settled in [21]. (See [32,8] for related results, and the discussion in [21].) Sharp crests in its periodic traveling wave of maximum amplitude was numerically supported in [17] and an analytical result was announced in [13]. (A complete proof of global bifurcation will be reported elsewhere.) In the last few years, the Whitham equation for water waves received renewed attention from a number of other vantage points; see [16,15,22,34,14], among others.

Benjamin and Feir in [3] and, independently, Whitham in [37], on the other hand, formally argued that a $2\pi/k$-periodic traveling wave in water under the influence of gravity would be unstable, leading to sidebands growth, namely the Benjamin–Feir or modulational instability, provided that $kd > 1.363\ldots$. Benny and Newell in [4] and Zakharov in [39], and later Hasimoto and Ono in [18] formally derived a nonlinear Schrödinger equation, which describes the long time behavior of the envelope of Stokes waves, and deduced the Benjamin–Feir instability. Kawahara in [28], Djordjević and Redekopp in [12] and later Hogan in [19] extended the results to capillary–gravity waves.

[1] The dispersion relation of water waves is well-known, whose derivation dates back to the work of Airy [1] in 1845!

[2] A relative error between c_{WW} and the phase speed for the KdV equation of, say, 10% is made for $kd > 1.242\ldots$.