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Weak maximum principles and geometric estimates for spacelike hypersurfaces in generalized Robertson–Walker spacetimes☆

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ABSTRACT

In this paper we deal with complete spacelike hypersurfaces in a generalized Robertson–Walker spacetime. Using as main analytical tool a new local form of the weak maximum principle for a class of operators including the Lorentzian mean curvature operator, we obtain some mean curvature estimates and height estimates for spacelike graphs with nice Bernstein type consequences. We also give height estimates for spacelike hypersurfaces with constant higher order mean curvature, which are obtained via the local form of the weak maximum principle recently given in Alías et al. (in press) for a class of operators including those constructed from the Newton tensors of a hypersurface.

1. Introduction

The study of constant mean curvature spacelike hypersurfaces in Lorentzian manifolds is motivated by physical and mathematical interests; indeed, Lorentzian geometry is the geometric language of general relativity. In this setting spacelike hypersurfaces play a central role being involved, for example, in the initial value formulation of the field equations. This latter consists, roughly speaking, in prescribing quantities determining the spacetime structure. To be more precise, one considers as initial data a triple (Σ, g, K), where Σ is a smooth manifold, g a Riemannian metric and K a symmetric tensor field on Σ, and one looks for a spacetime $(M, \langle , \rangle)$ satisfying Einstein's equation and possessing a spacelike hypersurface isometric to (Σ, g) and with extrinsic curvature (in physical language; that is, second fundamental form) K. This

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spacetime is foliated by spacelike hypersurfaces Σ_t that represent, in some sense, the time evolution of $\Sigma_0 := \Sigma$. See [21,14] for details and physical interpretations.

In 1968 Calabi showed that spacelike hypersurfaces in the Lorentz–Minkowski space $\mathbb{R}^{n+1}_1$ have a nice Bernstein-type property. More precisely in [9] he proved that the only complete maximal spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$, with $n \leq 4$, are spacelike hyperplanes, later on Cheng and Yau [12] extended the theorem to any dimension. In another direction, this result can be generalized to prove that spacelike hyperplanes are the only complete constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}_1$ with image of the Gauss map contained in a geodesic ball of the hyperbolic space (see [15,1,22]).

In this paper we deal with complete spacelike hypersurfaces in a generalized Robertson–Walker spacetime, that is, following the terminology introduced in [6], in a Lorentzian warped product manifold $M^{n+1} = -I \times_\varrho \mathbb{P}^n$. In the special case when the Riemannian factor $\mathbb{P}^n$ has constant sectional curvature we obtain the classical Robertson–Walker spacetimes. The warped manifold M^{n+1} is foliated by the slices $M_t := \{t\} \times \mathbb{P}^n$, that constitute a family of totally umbilical leaves with constant k-mean curvature, for each $k = 1, \dots, n$. Thus, the Calabi–Bernstein problem in this general setting becomes natural: it amounts to investigate under which circumstances a complete constant rth-mean curvature spacelike hypersurface is a spacelike slice M_t. In [6,3,13] the authors consider the compact case, that is the case of compact spacelike hypersurfaces in spatially closed general Robertson–Walker spacetimes. They show that, with some additional hypothesis on the ambient space, namely the null convergence condition, immersed compact spacelike hypersurfaces with constant k-mean curvature must be slices, unless other very special cases occur. Recall that a spacetime obeys the null convergence condition if its Ricci curvature is non-negative on lightlike directions. In [4] the authors extend this result to the complete case. To achieve their goal, they assume the sectional curvature of the Riemannian factor to be bounded from below and the hypersurface to be contained in a slab $[a, b] \times \mathbb{P}^n \subseteq M^{n+1}$, a condition automatically satisfied in the compact case.

An interesting approach to these uniqueness questions is via "a priori" height estimates for constant k-mean curvature spacelike hypersurfaces. Roughly speaking, estimates of this type give a quantitative measure of the deviation of the hypersurface from being a slice and they can also be used to get informations on the topology of the immersion at infinity.

Consider, now, a spacelike hypersurface $\Sigma \to -\mathbb{R} \times_\varrho \mathbb{P}^n$; call ∂_t the coordinate vector field along $\mathbb{R}$ and N the unique timelike normal globally defined on Σ with the same orientation of ∂_t. Define $\Theta := \langle N, \partial_t\rangle$. The assumption on Σ to be spacelike implies the bound $\Theta \leq -1$, that enables us to introduce the *hyperbolic angle* $\theta > 0$ defined by $\Theta = -\cosh\theta$. In some of our results we assume that Θ is bounded. Although this hypothesis seems only to have a technical role in our arguments, we show, by way of a simple counterexample, that it is indeed necessary. Furthermore, this assumption has a nice physical interpretation as follows. In a spacetime $(M, \langle , \rangle)$ the exponential map provides an efficient tool to formalize the concept of "observer". An observer on M is forced to move into the future, and thus it can be modeled by a future pointing timelike curve. Often it suffices to consider an instantaneous observer, that is, a couple (p, X), where p is a point of M and $X \in T_pM$ is a future pointing timelike direction. Along Σ we can consider two relevant observers: (p, N_p) and $(p, \partial_{t|_p})$. Consider the orthogonal decomposition $N = -\Theta\partial_t + w$, where $w \in T\mathbb{P}^n$. By definition $v := |w|/|\Theta| = \tanh\theta$ is the Newtonian speed of N observed by ∂_t. So, the assumption of the boundedness of θ assures that v does not approach the speed of light in vacuum. See [19, pp. 41–45] for further details.

The paper is organized as follows. In Section 2 we present our main analytical tool to study the problem, that is, a local form of the weak maximum principle in a Riemannian manifold $(M, \langle , \rangle)$ for a class of operators including the Lorentzian mean curvature operator. This version of the maximum principle, called the *open weak maximum principle*, has been introduced in [5] for operators of the form

$$L_\varphi u = \operatorname{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u),$$