Nonlinear Analysis 129 (2015) 160–172

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Second order estimates for boundary blow-up solutions of elliptic equations with an additive gradient term

Ester Giarrusso[a,*], Giovanni Porru[b]

[a] Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", University of Naples Federico II, MSA, Naples, Italy
[b] Dipartimento di Matematica e Informatica, University of Cagliari, Cagliari, Italy

ARTICLE INFO

Article history:
Received 3 June 2015
Accepted 8 September 2015
Communicated by Enzo Mitidieri

MSC:
35J25
35J62
35J67
35B40
35B44

Keywords:
Blow-up solutions
Quasilinear elliptic equations with additive gradient term
Second order effect

ABSTRACT

Let $\Omega \subset R^N$ be a bounded smooth domain and let $0 \leq \ell \leq 1$, $p > 1$. We investigate the effect of the mean curvature of the boundary $\partial\Omega$ on the behavior of the solution to the problem $\Delta u - \ell|Du|^{\frac{2p}{p+1}} = u^p$ in Ω, $u = \infty$ on $\partial\Omega$. We find an asymptotic expansion up to the second order of the solution u in terms of the distance from x to the boundary $\partial\Omega$.

1. Introduction

We study the boundary blowup problem

$$\triangle u - \ell|Du|^{\frac{2p}{p+1}} = u^p \quad \text{in } \Omega,\ u \to \infty \text{ as } x \to \partial\Omega, \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, $0 \leq \ell \leq 1$ and $p > 1$. We are interested in the behavior of the solution u near the boundary $\partial\Omega$. These solutions are called "blowup" or "large" solutions.

It is well known that the first order approximation of a large solution to the equation $\triangle u = u^p$, $p > 1$, depends only on the function $\delta(x) = \text{dist}(x, \partial\Omega)$ (it does not depend on the geometry of the domain Ω). In

* Corresponding author.
E-mail addresses: ester.giarrusso@unina.it (E. Giarrusso), porru@unica.it (G. Porru).

http://dx.doi.org/10.1016/j.na.2015.09.008
0362-546X/

fact it turns out that, near the boundary, [6,17]

$$u(x) = \left(\frac{p-1}{\sqrt{2(p+1)}}\delta(x)\right)^{\frac{2}{1-p}} \big(1+o(1)\big), \tag{2}$$

where $o(1) \to 0$ as $\delta(x) \to 0$.

Lately, for $p > 3$ this result was improved by Lazer and McKenna [16], who proved that

$$u(x) = \left(\frac{p-1}{\sqrt{2(p+1)}}\delta(x)\right)^{\frac{2}{1-p}} + o(1).$$

A further deeper analysis on the behavior of $u(x)$, carried out in [10] for $p < 3$ and in [2,3,7,8] for $p > 1$, shows that the second order effect in the expansion of $u(x)$ near the boundary depends on the mean curvature $H(\overline{x})$ of $\partial\Omega$ at the nearest point $\overline{x}$ to x. Namely, it results:

$$u(x) = \left(\frac{p-1}{\sqrt{2(p+1)}}\delta(x)\right)^{\frac{2}{1-p}} \left[1 + \frac{(N-1)H(\overline{x})}{p+3}\delta(x) + o(\delta)\right]$$

as $x \to \partial\Omega$.

The previous results were extended also to more general non linearities of the right hand side and more general nonlinear elliptic operators (see, for example [1,5,8,9,12,15,18–20,22,23] and the survey paper [21]).

As for the equation $\Delta u = u^p|Du|^q$ with $0 \le q < (p+3)/(p+2)$, in [13] it was proved the following expansion:

$$u(x) = \varphi(\delta(x)) \left[1 + \frac{(2-q)(N-1)H(\overline{x})}{2(p+3-q(p+2))}\delta(x) + o(\delta)\right],$$

where

$$\varphi(t) = \left(\frac{2-q}{p+q-1}\right)^{\frac{2-q}{p+q-1}} \left(\frac{p+1}{2-q}\right)^{\frac{1}{p+q-1}} t^{\frac{q-2}{p+q-1}}.$$

The problem

$$\Delta u - |Du|^q = f(u) \quad \text{in } \Omega,\ u(x) \to \infty \text{ as } x \to \partial\Omega,$$

where $f(u)$ is increasing, unbounded and smooth, was studied in [4]. The authors showed how the main asymptotic behavior of u is strongly influenced by the additive term $-|Du|^q$. For instance, when $f(u) = u^p$, (2) is still valid if $p > 1$ and $q < \frac{2p}{p+1}$; on the contrary, if $\max\left\{1, \frac{2p}{p+1}\right\} < q < 2$ and $p > 0$ then near the boundary

$$u(x) = \frac{1}{2-q}\left[(q-1)\delta(x)\right]^{\frac{q-2}{q-1}} (1+o(1)).$$

For the special case (1) we derive from the results in [11] that

$$u(x) = L(\delta(x))^{\frac{2}{1-p}}(1+o(1)),$$

where L is the solution of the equation

$$\ell\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} L^{\frac{p-1}{p+1}} + L^{p-1} = \frac{2(p+1)}{(p-1)^2}. \tag{3}$$

Actually, in [11] $\ell = 1$, but slightly modifying the proof therein one can take $\ell \in (0,1)$. In some sense, we see from calculations that, when $q = 2p/(p+1)$ and $p > 1$, the term $|Du|^{\frac{2p}{p+1}}$ is "equivalent" to the term u^p.