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Higher regularity of the free boundary in the elliptic Signorini problem

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ABSTRACT

In this paper we study the higher regularity of the free boundary for the elliptic Signorini problem. By using a partial hodograph–Legendre transformation we show that the regular part of the free boundary is real analytic. The first complication in the study is the invertibility of the hodograph transform (which is only $C^{0,1/2}$) which can be overcome by studying the precise asymptotic behavior of the solutions near regular free boundary points. The second and main complication in the study is that the equation satisfied by the Legendre transform is degenerate. However, the equation has a subelliptic structure and can be viewed as a perturbation of the Baouendi–Grushin operator. By using the L^p theory available for that operator, we can bootstrap the regularity of the Legendre transform up to real analyticity, which implies the real analyticity of the free boundary.

1. Introduction

1.1. Problem set-up and known results

Let B_R be the Euclidean ball in $\mathbb{R}^n$ $(n \geq 3)$ centered at the origin with radius $R > 0$. Let $B_R^+ = B_R \cap \{x_n > 0\}$, $B_R' = B_R \cap \{x_n = 0\}$ and $(\partial B_R)^+ = \partial B_R \cap \{x_n > 0\}$. Consider local minimizers of the Dirichlet functional

$$J(v) = \int_{B_R^+} |\nabla v|^2 dx$$

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over the closed convex set

$$\mathcal{K} = \{v \in W^{1,2}(B_R^+) : v \geq 0 \text{ on } B_R'\},$$

i.e. functions $u \in \mathcal{K}$ which satisfy

$$J(u) \leq J(v), \quad \text{for any } v \in \mathcal{K} \text{ with } v - u = 0 \text{ on } (\partial B_R)^+.$$

This problem is known as the *(boundary) thin obstacle problem* or the *(elliptic) Signorini problem*. It was shown in [2] that the local minimizers u are of class $C^{1,1/2}_{\mathrm{loc}}(B_R^+ \cup B_R')$. Besides, u will satisfy

$$\Delta u = 0 \quad \text{in } B_R^+, \tag{1.1}$$

$$u \geq 0, \ -\partial_{x_n} u \geq 0, \quad u \cdot \partial_{x_n} u = 0 \quad \text{on } B_R'. \tag{1.2}$$

The boundary condition (1.2) is known as the *complementarity or Signorini boundary condition*. One of the main features of the problem is that the following sets are a priori unknown:

$$\begin{aligned} \Lambda_u &= \{x \in B_R' : u(x) = 0\} && \textit{coincidence set} \\ \Omega_u &= \{x \in B_R' : u(x) > 0\} && \textit{positivity set} \\ \Gamma_u &= \partial_{B_R'} \Omega_u, && \textit{free boundary} \end{aligned}$$

where by $\partial_{B_R'}$ we understand the boundary in the relative topology of B_R'. The free boundary Γ_u sometimes is said to be *thin*, to indicate that it is (expected to be) of codimension two. One of the most interesting questions in this problem is the study of the structure and the regularity of the free boundary Γ_u.

To put our results in a proper perspective, below we give a brief overview of some of the known results in the literature. The proofs can be found in [3,4,14] and in Chapter 9 of [21].

We start by noting that we can extend solutions u of the Signorini problem (1.1)–(1.2) to the entire ball B_R in two different ways: either by even symmetry in x_n variable or by odd symmetry. The even extension will be harmonic in $B_R \setminus \Lambda_u$, while the odd extension will be so in $B_R \setminus \overline{\Omega_u}$. In a sense, those two extensions can be viewed as two different branches of a two-valued harmonic function. This gives a heuristic explanation for the monotonicity of Almgren's *frequency function*

$$N_{x_0}(u, r) = \frac{r \int_{B_r^+(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)^+} u^2},$$

which goes back to Almgren's study of multi-valued harmonic functions [1]. In particular, the limiting value

$$\kappa(x_0) = N_{x_0}(0_+, u)$$

for $x_0 \in \Gamma_u$ turns out to be a very effective tool in classifying free boundary points. By using the monotonicity of the frequency N_{x_0}, it can be shown that the rescalings

$$\tilde{u}_{r,x_0}(x) = \frac{u(x_0 + rx)}{\left(\fint_{\partial B_r^+(x_0)} u^2 \right)^{1/2}} \tag{1.3}$$

converge, over subsequences $r = r_j \to 0_+$, to solutions $\tilde{u}_0$ of the Signorini problem (1.1)–(1.2) in $\mathbb{R}^n_+$. Such limits are known as *blowups* of u at x_0. Moreover, it can be shown that such blowups will be homogeneous of degree $\kappa(x_0)$, regardless of the sequence $r_j \to 0_+$. It is readily seen from the definition that the mapping $x_0 \mapsto \kappa(x_0)$ is upper semicontinuous on Γ_u. Furthermore, it can be shown that $\kappa(x_0) \geq 3/2$ for every $x_0 \in \Gamma_u$ and, more precisely, that the following alternative holds:

$$\kappa(x_0) = 3/2 \quad \text{or} \quad \kappa(x_0) \geq 2.$$