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Self-dual solutions to pseudo Yang–Mills equations

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ABSTRACT

We study pseudo Yang–Mills fields on a compact 5-dimensional strictly pseudoconvex CR manifold M i.e. critical points to the functional $\mathcal{YM}_b(D) = \frac{1}{2}\int_M \left\|\Pi_H R^D\right\|^2 \theta\wedge(d\theta)^2$ on the space $\mathcal{C}(E,h)$ of all connections D on a Hermitian vector bundle $(E,\,h)$ over M, such that $Dh = 0$. If $\mathcal{A} = \{D \in \mathcal{C}(E,h) : \xi\rfloor R^D = 0,\ G^*_\theta(\mathrm{Tr}(R^D),\, d\theta) = 0\}$ and $D \in \mathcal{A}$ is an absolute minimum to $\mathcal{YM}_b : \mathcal{A} \to \mathbb{R}$ then (i) $\Delta_b \mathrm{Tr}(R^D) = 0$ and (ii) D is self-dual or anti-self-dual according to the sign of $c_2(\theta,D) = \int_M \theta\wedge\{\mathbf{P}_2(D) - \frac{m-1}{2m}\mathbf{P}_1(D)\wedge\mathbf{P}_1(D)\}$ [where $\mathbf{P}_k(D)$ is the k-th Chern form of (E,D)] and provided $c_2(\theta,D)$ is constant on $\mathcal{A}$.

1. Introduction

Let $(E,\,h) \to M$ be a Hermitian vector bundle over a compact 5-dimensional CR manifold $(M, T_{1,0}(M))$ of CR dimension 2 and let θ be a positively oriented contact form on M. The pseudo Yang–Mills functional (cf. [1]) is

$$\mathcal{YM}_b(D) = \frac{1}{2}\int_M \left\|\Pi_H R^D\right\|^2\ \Psi_\theta, \quad D \in \mathcal{C}(E,h), \tag{1}$$

where $\Psi_\theta = \theta\wedge(d\theta)^2$ and $R^D \in \Omega^2(\mathrm{Ad}\,E)$ is the curvature of D. Also $\Pi_H : \Omega^2(\mathrm{Ad}\,E) \to \Omega^2(\mathrm{Ad}\,E)/\mathcal{J}^2_\theta(\mathrm{Ad}\,E)$ is the natural projection and $\mathcal{J}^\bullet_\theta(\mathrm{Ad}\,E)$ is the ideal in $\Omega^\bullet(\mathrm{Ad}\,E)$ generated by θ. The functional (1) has been discovered in [1] by integrating along the fibers in the ordinary Yang–Mills functional, built with respect to

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$\pi^* h$ and the Fefferman metric F_θ, on the pullback bundle $\pi^* E \to C(M)$ over the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$.

Let $\mathcal{C}(E,h)$ be the affine space of all connections D in E which parallelize h. A connection $D \in \mathcal{C}(E,h)$ is a *pseudo Yang–Mills field* on (M,θ) if

$$\frac{d}{dt}\left\{\mathcal{YM}_b\left(D + t\varphi\right)\right\}_{t=0} = 0 \tag{2}$$

for any $\varphi \in \Omega^0(\mathrm{Ad}\, E)$. The corresponding Euler–Lagrange equations are shown to be (cf. Section 4)

$$\delta_D^H \, \Pi_H R^D = 0, \qquad \langle R^D, \, S \otimes d\theta \rangle = 0, \tag{3}$$

for any $S \in \Omega^0(\mathrm{Ad}\, E)$, where $\delta_D^H = \left([d_D^{(1)}]^H\right)^* : \Omega_H^2(\mathrm{Ad}\, E) \to \Omega_H^1(\mathrm{Ad}\, E)$ is the formal adjoint of $[d_D^{(1)}]^H : \Omega_H^1(\mathrm{Ad}\, E) \to \Omega_H^2(\mathrm{Ad}\, E)$. Here we have set

$$\Omega_H^k(\mathrm{Ad}\, E) = \left\{\omega \in \Omega^k(\mathrm{Ad}\, E) : \xi \rfloor \, \omega = 0\right\}$$

and $\xi \in \mathfrak{X}(M)$ is the Reeb vector of (M,θ).

Self-dual or anti-self-dual $\mathrm{SU}(m)$ connections in Hermitian vector bundles E over compact oriented 4-dimensional Riemannian manifolds M are absolute minima to the Yang–Mills functional (cf. e.g. Theorem 3.2.3 in [13, p. 123]) according to the sign of $c_2(E)[M]$, where $c_2(E)$ is the second Chern class of E. In the case considered in this work $\dim(M) = 5$ yet the maximally complex distribution $H(M) = \mathrm{Re}\left\{T_{1,0}(M) \oplus T_{0,1}(M)\right\}$ has real rank 4 hence the Hodge operator $*_H : \Lambda^2 H(M)^* \to \Lambda^2 H(M)^*$ (associated to the Levi form G_θ) squares to $*_H^2 = \mathrm{Id}$. Consequently (cf. [11,12])

$$\Lambda^2 H(M)^* = \Lambda_+^2 H(M)^* \oplus \Lambda_-^2 H(M)^* \tag{4}$$

where $\Lambda_\pm^2 H(M)^* = \mathrm{Eigen}\left(*_H; \pm 1\right)$. The decomposition (4) prompts the natural generalization of (anti) self-duality i.e. a connection $D \in \mathcal{C}(E,h)$ is $\pm$ *self-dual* (*self-dual* or *anti-self-dual*) if

$$\Pi_H R^D \in C^\infty(\Lambda_\pm^2 H(M)^* \otimes \mathrm{Ad}(E)).$$

We show (cf. Theorem 2 in Section 7) that a self-dual (respectively anti-self-dual) connection $D \in \mathcal{C}(E,h)$ is a pseudo Yang–Mills field if and only if its curvature R^D is horizontal (respectively R^D is horizontal and $\Lambda_\theta R^D = 0$). Let us set

$$\mathcal{A} = \left\{D \in \mathcal{C}(E,h) : \xi \rfloor \, R^D = 0, \; G_\theta^*\left(\mathrm{Tr}(R^D), \, d\theta\right) = 0\right\}$$

and consider the problem $\min_{D \in \mathcal{A}} \mathcal{YM}_b(D)$. This is shown to be equivalent to the problem

$$\min_{D \in \mathcal{A}} \int_M \left\|\mathrm{Tr}\left(R^D\right)\right\|^2 \, \Psi_\theta, \qquad \min_{D \in \mathcal{A}} \left\{x_D + y_D\right\},$$
$$x_D \equiv \int_M \left\|\left(R_0^D\right)_+\right\|^2 * 1, \qquad y_D \equiv \int_M \left\|\left(R_0^D\right)_-\right\|^2 * 1,$$

($\Psi_\theta = c_{0,2} * 1$). On the other hand we show (cf. Section 8) that

$$-x_D + y_D = 8\pi^2 \, c(\theta, D) \tag{5}$$

where

$$c(\theta, D) = \int_M \theta \wedge \left\{\mathbf{P_2}(D) - \frac{m-1}{2m}\, \mathbf{P}_1(D) \wedge \mathbf{P}_1(D)\right\}$$

and $\mathbf{P}_k(D) = P_k\left(\frac{i}{2\pi} R^D\right)$ is the k-th Chern form of (E,D). The counterpart to (5) on a compact 4-dimensional Riemannian manifold is the relation

$$-x_D + y_D = 8\pi^2 \left\{c_2(E) - \frac{m-1}{2m}\, c_1(E)^2\right\}[M] \tag{6}$$