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Subelliptic Peter–Weyl and Plancherel theorems on compact, connected, semisimple Lie groups

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A R T I C L E I N F O

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A B S T R A C T

We will study the connections between the elliptic and subelliptic versions of the Peter–Weyl and Plancherel theorems, in the case when the sub-Riemannian structure is generated naturally by the choice of a Cartan subalgebra. Along the way we will introduce and study the subelliptic Casimir operator associated to the subelliptic Laplacian.

1. Introduction

In this paper we will show that the subelliptic spectral analysis results needed to study the subelliptic heat kernel and to obtain precise bounds of the L^2-norms of the non-horizontal directional derivatives, have natural and simple expressions in the case of compact and semi-simple Lie groups. The key ingredient is the connection between the elliptic and subelliptic versions of the Peter–Weyl and Plancherel theorems, when the subelliptic structure is generated by the orthogonal complement of a maximal commutative subalgebra of the Lie algebra.

Studies regarding the subelliptic heat kernel started immediately after Hörmander's paper [13] appeared. Initially these studies were focused mostly on the nilpotent Lie groups [12], with a more recent shift toward non-nilpotent Lie groups [1–5], followed independently, and somewhat logically, by parallel studies on regularity of nonlinear subelliptic PDE's [7,8,6].

Let $\mathbb{G}$ be a compact, connected, semi-simple matrix Lie group and $\mathcal{G}$ its Lie algebra. In this context, working with matrix groups is not a restriction, but has the advantage of an easy setup for the formulas we will use. We denote by $\mathcal{M}_n(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{C})$ the vector space of $n \times n$ real or complex matrices and by $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ the Lie groups of the invertible $n \times n$ matrices. Matrix groups are defined as closed subgroups of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ and hence inherit the Lie group structure.

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The Lie algebra of $\mathbb{G}$ can be defined by using the matrix exponential:

$$\mathcal{G} = \{X \in \mathcal{M}_n(\mathbb{R} \text{ or } \mathbb{C} \text{ depending on } \mathbb{G}) \mid \exp(tX) \in \mathbb{G},\ \forall\, t \in \mathbb{R}\}.$$

From this definition it follows that $\mathcal{G}$ is a real vector space and contains the generators of the 1-parameter subgroups of $\mathbb{G}$. $\mathcal{G}$ becomes an algebra when it is endowed with the bilinear operator,

$$[X, Y] = XY - YX,$$

called the commutator of X and Y, which measures the non-commutativity at infinitesimal scales in $\mathbb{G}$. A commutative Lie algebra has the property that $[X, Y] = 0,\ \forall\ X, Y \in \mathcal{G}$ and a semi-simple Lie algebra is on the opposite end of the commutativity scale, as it cannot have any non-trivial commutative ideals. A Lie group is semi-simple if its Lie algebra is semi-simple.

The adjoint representation of $\mathbb{G}$ is the group homomorphism

$$\mathrm{Ad} : \mathbb{G} \to \mathrm{Aut}(\mathcal{G}), \quad \mathrm{Ad}x\,(X) = xXx^{-1},$$

while its differential at the identity is the Lie algebra homomorphism

$$\mathrm{ad} : \mathcal{G} \to \mathrm{End}(\mathcal{G}), \quad \mathrm{ad}X\,(Y) = [X, Y].$$

The Killing form

$$K(X, Y) = \mathrm{trace}\,(\mathrm{ad}X \cdot \mathrm{ad}Y),$$

is negative definite and non-degenerate on the Lie algebra of a compact, semi-simple Lie group, and hence we can define an inner product on $\mathcal{G}$ as

$$\langle X, Y\rangle = -\rho\, K(X, Y), \tag{1.1}$$

where $\rho > 0$ is a constant, which can be adjusted for each Lie algebra according to our normalization needs. The Killing form is Ad invariant, therefore $\mathrm{Ad}x$ is a unitary linear transformation for all $x \in G$ and $\mathrm{ad}X$ is skew-symmetric for all $X \in \mathcal{G}$.

We will consider a natural sub-Riemannian geometry on $\mathbb{G}$, which is defined by the choice of a maximal, commutative subalgebra of $\mathcal{G}$, called a Cartan subalgebra.

Let us fix a Cartan subalgebra Γ and let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be an orthonormal basis of it.

We extend the inner product of $\mathcal{G}$ bi-linearly to the complexified Lie algebra $\mathcal{G}_{\mathbb{C}} = \mathcal{G} \bigoplus i\mathcal{G}$. The mappings $\mathrm{ad}T : \mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}},\ T \in \Gamma$, commute and are skew-symmetric, so they can be simultaneously diagonalized and have purely imaginary eigenvalues.

We define $\alpha \in \Gamma$ to be a root if $\alpha \neq 0$ and $\mathcal{G}_\alpha \neq \{0\}$, where

$$\mathcal{G}_\alpha = \{Z \in \mathcal{G}_{\mathbb{C}} :\ \mathrm{ad}T(Z) = i\,\langle \alpha, T\rangle\, Z,\ \forall\, T \in \mathcal{T}\ \}. \tag{1.2}$$

Let $\mathcal{R}$ be the set of all roots, which will be ordered by the relation $\alpha > \beta$ if $\alpha - \beta$ has its first non-zero coordinate positive. We denote by $\mathcal{P}$ the set of all positive roots and let

$$\delta = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} \alpha.$$

For the most important properties of $\mathcal{G}_\alpha$ we quote [9,14]:

(i) $\dim_{\mathbb{C}} \mathcal{G}_\alpha = 1$.

(ii) $\mathcal{G}_0 = \Gamma_{\mathbb{C}}$.

(iii) $\mathcal{G}_{-\alpha} = \overline{\mathcal{G}_\alpha}$.

(iv) $\langle \mathcal{G}_\alpha, \mathcal{G}_\beta\rangle = 0 \quad \text{if } \beta \neq \pm\alpha$.