Nonlinear Analysis 126 (2015) 159–169

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

# Curvature-dimension estimates for the Laplace–Beltrami operator of a totally geodesic foliation

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A R T I C L E   I N F O

*Article history:*
Received 10 December 2014
Accepted 12 June 2015
Communicated by Enzo Mitidieri

*Keywords:*
Laplacian
Riemannian foliation
Curvature dimension inequalities

A B S T R A C T

We study Bakry-Émery type estimates for the Laplace–Beltrami operator of a totally geodesic foliation. In particular, we are interested in situations for which the $\Gamma_2$ operator may not be bounded from below but the horizontal Bakry-Émery curvature is. As we prove it, under a bracket generating condition, this weaker condition is enough to imply several functional inequalities for the heat semigroup including the Wang–Harnack inequality and the log-Sobolev inequality. We also prove that, under proper additional assumptions, the generalized curvature dimension inequality introduced by Baudoin and Garofalo (2015) is uniformly satisfied for a family of Riemannian metrics that converge to the sub-Riemannian one.



## 1. Introduction

In the recent few years, there have been several works using Riemannian geometry tools to study sub-Laplacians. We refer to the survey [4] for an overview of those techniques. In the present work we somehow take the opposite stance and show how sub-Riemannian geometry can be used to study Laplacians. More precisely, we shall be interested in the Laplace–Beltrami operator of a Riemannian foliation with totally geodesic leaves. Our main assumption will be that the horizontal distribution of the foliation is bracket-generating and the horizontal Bakry–Émery curvature of the Laplace–Beltrami operator is bounded from below. However, we will not assume anything on the vertical Bakry–Émery curvature. As a consequence the $\Gamma_2$ operator does not need to be bounded from below. Surprisingly, even under this weak condition, we are still able to obtain several important functional inequalities for the heat semigroup. We mention in particular the Wang–Harnack inequality and the associated criterion for the log-Sobolev inequality.

In the second part of our work, we show that the Baudoin–Garofalo generalized curvature dimension inequality [6] for the horizontal Laplacian of the foliation can be seen as a uniform limit of curvature

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http://dx.doi.org/10.1016/j.na.2015.06.025
0362-546X/

dimension estimates for the Laplace–Beltrami operator of the canonical variations of the metric of the foliation.

The paper is organized as follows. In Section 2 we introduce the geometric setting and establish Bochner's identities for the Laplace–Beltrami operator of a Riemannian foliation with totally geodesic leaves. These formulas are new and interesting in themselves. The main geometric novelty in those formulas is to separate the Bakry–Émery curvature of the Laplacian into two parts: an horizontal part and a vertical part.

In Section 3 we assume that the horizontal Bakry–Émery curvature is bounded from below and deduce several gradient bounds for the diffusion semigroup. We are able to prove the Wang–Harnack inequality and deduce from it a criterion for the log-Sobolev inequality.

In Section 4, we show that if additionally the vertical Bakry–Émery curvature is bounded from below, we get a uniform family of curvature-dimension estimates for a one-parameter of Laplacians. This curvature dimension inequalities interpolate between the classical Bakry–Émery curvature dimension condition and the Baudoin–Garofalo curvature dimension inequality introduced in [6].

## 2. Bochner identities for the Laplace–Beltrami operator of a totally geodesic foliation

The goal of the section will be to prove Bochner's type identities for the Laplace–Beltrami operator of a totally geodesic Riemannian foliation. We refer to [4] for a detailed account about the geometry of such foliations.

We consider a smooth $n+m$ dimensional connected manifold $\mathbb{M}$ which is equipped with a Riemannian foliation with a complete bundle like metric $g$ and totally geodesic leaves. We moreover assume that the metric $g$ is complete and that the horizontal distribution $\mathcal{H}$ of the foliation is Yang–Mills (see [4]). We shall also assume that $\mathcal{H}$ is bracket-generating.

The $m$ dimensional sub-bundle $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of *vertical directions*. The sub-bundle $\mathcal{H}$ which is normal to $\mathcal{V}$ is referred to as the set of *horizontal directions*. The metric $g$ can be split as

$$g = g_{\mathcal{H}} \oplus g_{\mathcal{V}},$$

and we introduce the one-parameter family of Riemannian metrics:

$$g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0.$$

It is called the canonical variation of $g$. The Riemannian distance associated with $g_\varepsilon$ will be denoted by $d_\varepsilon$. Finally we denote by $\mu_\varepsilon$ the Riemannian volume associated to $g_\varepsilon$.

The Laplace–Beltrami operator of the metric $g_\varepsilon$ is given by

$$\Delta_{\varepsilon} = \Delta_{\mathcal{H}} + \varepsilon \Delta_{\mathcal{V}},$$

where $\Delta_{\mathcal{H}}$ is the horizontal Laplacian of the foliation and $\Delta_{\mathcal{V}}$ the vertical Laplacian.

The Bott connection which is defined in terms of the Levi-Civita connection $D$ of the metric $g$ by

$$\nabla_X Y = \begin{cases} (D_X Y)_{\mathcal{H}}, & X, Y \in \Gamma^{\infty}(\mathcal{H}) \\ [X, Y]_{\mathcal{H}}, & X \in \Gamma^{\infty}(\mathcal{V}), Y \in \Gamma^{\infty}(\mathcal{H}) \\ [X, Y]_{\mathcal{V}}, & X \in \Gamma^{\infty}(\mathcal{H}), Y \in \Gamma^{\infty}(\mathcal{V}) \\ (D_X Y)_{\mathcal{V}}, & X, Y \in \Gamma^{\infty}(\mathcal{V}) \end{cases}$$

where the subscript $\mathcal{H}$ (resp. $\mathcal{V}$) denotes the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). Let us observe that for horizontal vector fields $X, Y$ the torsion $T(X, Y)$ is given by

$$T(X, Y) = -[X, Y]_{\mathcal{V}}.$$