Nonlinear Analysis 126 (2015) 218–233

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Area formula for centered Hausdorff measures in metric spaces

Bruno Franchi[a,*], Raul Paolo Serapioni[b], Francesco Serra Cassano[b]

[a] *Dipartimento di Matematica, Università di Bologna, Piazza di porta San Donato, 5, 40126 Bologna, Italy*
[b] *Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123, Povo (Trento), Italy*

ARTICLE INFO

Article history:
Received 13 December 2014
Accepted 10 February 2015
Communicated by Enzo Mitidieri

Keywords:
Hausdorff measures
Metric spaces
Perimeter
Carnot groups

ABSTRACT

Motivated by an example in Magnani (in press), we study, inside a separable metric space (X, d), the relations between centered and non centered m-dimensional densities of a Radon measure μ in X and their relations with spherical and centered spherical m-dimensional Hausdorff measures. Eventually we give an application to finite perimeter sets in Carnot groups.

1. Introduction

In a recent interesting note [15], Valentino Magnani observed the following fact. In a separable metric space (X, d), endowed with a Radon measure μ, absolutely continuous with respect to the m-dimensional spherical measure $\mathcal{S}^m$, the area formula for μ with respect to $\mathcal{S}^m$ i.e.

$$\mu(B) = \int_B \Theta_F^{*m}(\mu, x)\, d\mathcal{S}^m(x) \tag{1.1}$$

for any Borel set B may fail to be true if the m-dimensional Federer density $\Theta_F^{*m}(\mu, \cdot)$ is substituted by the (centered) m-dimensional density $\Theta^{*m}(\mu, \cdot)$ (see Definition 1.7(i) and (ii)).

Indeed Magnani provides the following example: in the Heisenberg group $X = \mathbb{H}^1 \equiv \mathbb{R}^3$, equipped with its sub-Riemannian metric d, there are a Radon measure μ, a set $A \subset \mathbb{H}^1$ and two constants $0 < k_1 < k_2$ such that μ is absolutely continuous w.r.t. $\mathcal{S}^2$ and for all $x \in A$

$$\Theta^2(\mu, x) = k_1 < k_2 = \Theta_F^{*2}(\mu, x)$$

and for all $t \in (k_1, k_2)$

$$\mu(A) > t\, \mathcal{S}^2(A). \tag{1.2}$$

* Corresponding author.
E-mail addresses: bruno.franchi@unibo.it (B. Franchi), serapion@science.unitn.it (R.P. Serapioni), cassano@science.unitn.it (F. Serra Cassano).

http://dx.doi.org/10.1016/j.na.2015.02.008
0362-546X/

Because of (1.2), given $A \subset X$ and $k > 0$, the implication

$$\Theta^m(\mu, x) = k \quad \forall x \in A \Rightarrow \mu \llcorner A = k\, \mathcal{S}^m \llcorner A \tag{1.3}$$

fails to be true in general.

Implication (1.3) was used by us to prove that the perimeter measure $|\partial E|_{\mathbb{G}}$ agrees, up to a multiplicative constant, with the $(Q-1)$-dimensional spherical Hausdorff measure $\mathcal{S}^{Q-1}$, in a step 2 Carnot group $\mathbb{G}$ of Hausdorff dimension Q. Hence a new proof of this result is in order.

Indeed, in [14], Magnani himself provides an alternative proof of our result using his new notion of $(n-1)$ *vertical regular* distance (see Theorem 4.19 in this paper).

We take here a different approach to the same topic. From the preceding considerations it appears that Federer density $\Theta_F^{*m}(\mu, x)$ plays a privileged role in area formulas when the spherical measure $\mathcal{S}^m$ is used. On the other hand, non centered densities as $\Theta_F^{*m}(\mu, x)$ are often harder to compute than the corresponding centered densities $\Theta^{*m}(\mu, x)$. Therefore, motivated by Magnani's note, we looked for an area formula different from (1.1) in which the density $\Theta^{*m}(\mu, x)$ is used, but the measure $\mathcal{S}^m$ is replaced by an equivalent one.

Centered Hausdorff measures $\mathcal{C}^m$ (see Definition 2.1(iii)) seem to be the right substitutes. Indeed we could prove the following theoretic area formula: if A is a Borel set in a metric space X, if $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m \llcorner A$ then for each Borel $B \subset A$,

$$\mu(B) = \int_B \Theta^{*m}(\mu, x)\, d\mathcal{C}^m(x), \tag{1.4}$$

see Theorem 3.1 and Corollary 3.14.

Centered Hausdorff measures $\mathcal{C}^m$ were introduced in [19] to estimate more efficiently the Hausdorff dimension of self-similar fractal sets (see also [13]). Inside a general metric space a detailed study of centered Hausdorff measures has been carried on in [6].

Spherical and centered Hausdorff measures $\mathcal{S}^m$ and $\mathcal{C}^m$ may disagree (see [19]), even if they are equivalent, that is

$$\mathcal{S}^m \le \mathcal{C}^m \le 2^m\, \mathcal{S}^m.$$

However in the Euclidean case, i.e. when $X = \mathbb{R}^n$, they agree on *rectifiable* sets (see [19]). We show that this coincidence keeps being true for the simplest 1-dimensional submanifolds, namely Lipschitz curves, within a general metric setting (see Theorem 2.6). Besides, the coincidence is still true inside Carnot groups for the case of homogeneous dimension Q (see Corollary 4.13) and for 1-codimensional *intrinsic rectifiable* sets (see Theorem 4.28).

Using area formula (1.4) a new proof of the previously mentioned representation result for the perimeter measure $|\partial E|_{\mathbb{G}}$ follows, so filling – in a different way – the gaps in [8,9,16].

Let us introduce some notation and notions. Throughout this paper (X, d) is a separable metric space,

$$B(a, r) := \{x \in X : d(a, x) \le r\}$$

are the *closed ball* with center a and radius $r > 0$. The *diameter* of a set $E \subset X$ is denoted as

$$\operatorname{diam}(E) := \sup\{d(x, y) : x, y \in E\}.$$

If μ is an outer measure in X and $A \subset X$ the *restriction* of μ to A is denoted as

$$\mu \llcorner A(E) = \mu(A \cap E) \quad \text{if } E \subset X.$$

We assume the following condition on the diameter of closed balls: *there are constants* ρ_0, $0 < \rho_0 \le 2$ *and* $\delta_0 > 0$ *such that, for all* $r \in (0, \delta_0)$ *and* $x \in X$,

$$\operatorname{diam}(B(x, r)) = \rho_0\, r. \tag{1.5}$$