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Interior $W_*^{1,p(x,t)}$-regularity for parabolic divergence equations of Hörmander's type☆

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A B S T R A C T

Let $X_1, X_2, \ldots, X_q$ be a system of real smooth vector fields satisfying Hörmander's condition in a bounded domain $\Omega \subset \mathbb{R}^n$. We consider the following parabolic equations

$$u_t + X_i^* (a_{ij}(x,t) X_j u) = X_i^* f_i \quad \text{in } \Omega_T,$$

where X_i^* is the formal adjoint of X_i, the coefficients $a_{ij}(x,t)$ are real valued measurable functions defined in $\Omega_T = \Omega \times (0, T]$, satisfying the uniform parabolic condition

$$\mu |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x,t) \xi_i \xi_j \le \mu^{-1} |\xi|^2, \tag{0.1}$$

for almost every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, every $\xi \in \mathbb{R}^q$ and some constant μ. We prove the interior $W_*^{1,p(x,t)}$-regularity for weak solutions to the parabolic equations, under the assumptions that $p(x,t)$ satisfies the strong log-Hölder continuity condition and the coefficients $a_{ij}(x,t)$ belong to the space $VMO_{loc} \cap L^\infty$. Our method relies on a Gagliardo–Nirenberg inequality constituted on Hörmander's vector fields and a certain Vitali covering lemma.

1. Introduction

Let $X_i = \sum_{j=1}^n b_{ij}(x) \partial_{x_j}$, $i = 0, 1, 2, \ldots, q$ be a family of real smooth vector fields defined in a neighborhood of some bounded domain $\Omega \subset \mathbb{R}^n$ ($q \le n$), satisfying Hörmander's condition: the Lie algebra

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generated by $X_0, X_1, \ldots, X_q$ at any point of Ω spans $\mathbb{R}^n$ $(n \geq 3)$. Let

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2 + X_0.$$

Since Hörmander in [22] proved that the operator $\mathcal{L}$ is hypoelliptic, regularity for operators or equations constituted on Hörmander's vector fields has been widely studied (see [18,4,5,24,25,21,2,34,6,31,35]). Let us mention some of them. In '75, by assuming the existence of an underlying structure of homogeneous group for the operator $\mathcal{L}$, Folland in [18] established a priori L^p estimates for operator $\mathcal{L}$, that is, for any test function u, the L^p norm of X_iX_ju can be controlled by the L^p norm of Lu and u. Rothschild and Stein in [31] extended Folland's result to any system of smooth Hörmander's vector fields, by using the lifting and approximation technique. Later, L^p estimates for the operators with VMO_{loc} coefficients, such as

$$\mathcal{L} = \sum_{i=1}^{q} a_{ij}(x) X_iX_j \quad \text{and} \quad \mathcal{L} = \sum_{i=1}^{q} a_{ij}(x) X_iX_i + X_0,$$

were studied in [4,6] by exploiting the representation method of Chiarenza–Frasca–Longo [11], respectively. The same technique can also be used to obtain the L^p regularity for divergence equations with VMO coefficients (see [17,35]).

From the aforementioned literature, we see that the regularity for partial differential equations can be well characterized in the frame of the Lebesgue spaces L^p. However, if one wishes to investigate the materials with inhomogeneities such as electrorheological fluids, this is not enough, since in this special case, the exponent p is expected to be able to vary. In view of this, the variable exponent Lebesgue spaces were introduced in the studying of regularity for PDES (see [16]). The concept of variable exponent Lebesgue spaces was first introduced by W. Orlicz [30] in 1931. After the pioneering works of H. Nakano [28,29] and Diening [14], the basic properties of harmonic analysis on the variable exponent Lebesgue spaces, such as the singular integrals, maximal functions and Sobolev embedding were extensive studied, we refer the readers to the book [15], and references therein.

Recently, the authors in [8] obtained the $W^{1,p(\cdot)}$-regularity for divergence Dirichlet problem

$$\begin{cases} \operatorname{div}(a_{ij}(x) Du) = \operatorname{div} \mathbf{F}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p(\cdot)$ is a variable exponent, the coefficients a_{ij} have partially small BMO semi-norms and the boundary of the domain is Reifenberg flat. The proof in [8] is relied on the compactness method, which has been designed and exploited in a series of papers by Byun, Jia and Wang to deal with elliptic or parabolic equations in Reifenberg flat domains, see [7,9,32] and the references therein. We must emphasize that the compactness method is a powerful analysis tool, it can also be used to investigate the regularity for subelliptic equations or systems, for instances, see [34,35,33].

The paper is devoted to studying the following parabolic equations of divergence form

$$u_t + X_i^*(a_{ij}(x,t) X_ju) = X_i^* f_i, \quad (x,t) \in \Omega_T, \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T]$, X_i's satisfy the Hörmander's condition in Ω, and X_i^* is the formal adjoint of X_i. We are interested in the interior $W_*^{1,p(x,t)}$-regularity for the weak solutions to (1.1). More precisely, we will prove that if $Q' \Subset \Omega_T$ and $f_i \in L^{p(\cdot)}(\Omega_T)(i = 1, \ldots, q)$, then $u \in W_*^{1,p(\cdot)}(Q')$, under the assumptions that $p(\cdot) = p(x,t)$ satisfies the strong log-Hölder continuity condition in Ω_T (see Section 2), and the coefficients $a_{ij}(x,t)$ belong to the class VMO_{loc} defined with respect to the subelliptic metric induced by the vector fields. Our method relies on a Gagliardo–Nirenberg inequality of Hörmander's type, the approximation result proved in [33] and a certain Vitali type covering lemma. To the best of our knowledge, our result is even new for the corresponding Euclidean case.