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Regularity of mean curvature flow of graphs on Lie groups free up to step 2

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ABSTRACT

We consider (smooth) solutions of the mean curvature flow of graphs over bounded domains in a Lie group free up to step two (and not necessarily nilpotent), endowed with a one parameter family of Riemannian metrics σ_ϵ collapsing to a subRiemannian metric σ_0 as $\epsilon \to 0$. We establish $C^{k,\alpha}$ estimates for this flow, that are uniform as $\epsilon \to 0$ and as a consequence prove long time existence for the subRiemannian mean curvature flow of the graph. Our proof extend to the setting of every step two Carnot group (not necessarily free) and can be adapted following our previous work in Capogna et al. (2013) to the total variation flow.

1. Introduction

The mean curvature flow is the motion of a surface where each point is moving in the direction of the normal with speed equal to the mean curvature. In the case where the evolution of graphs $S_t = \{(x, u(x,t))\} \subset \mathbb{R}^n \times \mathbb{R}$ is considered, then, provided enough regularity is assumed, the function u satisfies the equation

$$\partial_t u = \sqrt{1+|\nabla u|^2}\, div \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right).$$

Given appropriate boundary/initial conditions, global in time solutions asymptotically converge to minimal graphs.

In this paper we study long time existence of graph solutions of the mean curvature flow in a special class of degenerate Riemannian ambient spaces: The setting of sub-Riemannian manifolds [21,37]. In particular we will focus on a class of Lie groups endowed with a metric structure (G, σ_0) that arises as limit of collapsing left-invariant Riemannian structures (G, σ_ϵ).

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Our approach to the existence of global (in time) smooth solutions is based on a Riemannian approximation scheme. We study graph solutions of the mean curvature flow in the Riemannian spaces (G, σ_ϵ) where G is a group and σ_ϵ is a family of Riemannian metrics that 'collapse' as $\epsilon \to 0$ to a sub-Riemannian metric σ_0 in G.

Our results are analogue to those we proved for the total variation flow proved in [9]. The main difference is that we remove here the assumption that the group G is a Carnot group, i.e. we also consider non-nilpotent groups such as the group of rigid Euclidean motions $\mathcal{RT}$. In fact the results in the present paper yield, with minor modifications of the proof, the regularity and long time existence of the total variation flow in the same extended class of groups. The main new technical challenge that distinguishes the study of the total variation flow from the mean curvature flow is that the equation studied here is not in divergence form, which makes it necessary to completely change the proof of the $C^{1,\alpha}$ regularity.

1.1. Lie group structure

Let G be an analytic and simply connected Lie group with topological dimension n.

A subRiemannian manifold on G is a triplet $(G; \Delta; \sigma_0)$ where Δ denotes a left invariant bracket-generating subbundle of TM, and σ_0 is a positive definite smooth, bilinear form on Δ, (see for instance Montgomery [30]). We fix a orthonormal horizontal basis $X_1, \ldots, X_m$ of Δ. We will say that the group has step 2 if $\{X_i\}_{i=1,\ldots,m} \cup \{[X_i, X_j]\}_{i,j=1,\ldots,m}$ span the whole tangent space at every point. If in addition the vector fields

$$\{X_i\}_{i=1,\ldots,m} \quad \text{and} \quad \{[X_i, X_j]\}_{i,j=1,\ldots,m}$$

are linearly independent we say that the group is free up to step 2. We can complete $X_1, \ldots, X_m$ to a basis $(X_1, \ldots, X_n)$ of the Lie algebra $\mathcal{G}$, choosing a basis of the second layer of the tangent space. We denote by $(X_1, \ldots, X_n)$ (resp. $(X_1^r, \ldots, X_n^r)$) the left invariant (resp. right invariant) translations of the frames $(X_1, \ldots, X_n)$. We will say that the vector fields $X_1, \ldots, X_m$ have degree 1 and denote $d(X_i) = 1$ while their commutators of the form $[X_i, X_j]$ have degree 2. Throughout the paper we will assume for simplicity that the horizontal frame above is self-adjoint, the results continue to hold with minimal modifications of the proofs if this hypothesis is not satisfied.

As prototypes for this class of spaces we highlight the following:

- The standard example for such families is the Heisenberg group $\mathbb{H}^1$. This is a Lie group whose underlying manifold is $\mathbb{R}^3$ and is endowed with a group law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - (x_2 y_1 - x_1 y_2)).$$

With respect to such law one has that the vector fields

$$X_1 = \partial_{x_1} - x_2 \partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3}$$

are left-invariant. Together with their commutator $[X_1, X_2] = 2\partial_{x_3}$ they yield a basis of $\mathbb{R}^3$.
- A second example is given by the classical group of rigid motions of the plane, also known as the *roto-translation* group $\mathcal{RT}$. This is a Lie group with underlying manifold $\mathbb{R}^2 \times S^1$ and a group law $(x_1, x_2, \theta_1)(y_1, y_2, \theta_2) = (x_1 + y_1 \cos\theta - y_2 \sin\theta, x_2 + y_1 \sin\theta + y_2 \cos\theta, \theta_1 + \theta_2)$. The horizontal distribution is given by

$$\Delta = span\{X_1, X_2\}, \quad \text{with } X_1 = \cos\theta \partial_{x_1} + \sin\theta \partial_{x_2}, \text{ and } X_2 = \partial_\theta.$$