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Decay properties of solutions to the Cauchy problem for the scalar conservation law with nonlinearly degenerate viscosity

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A B S T R A C T

In this paper, we study the decay rate in time to solutions of the Cauchy problem for the one-dimensional viscous conservation law where the far field states are prescribed. Especially, we deal with the case that the flux function which is convex and also the viscosity is a nonlinearly degenerate one (p-Laplacian type viscosity). As the corresponding Riemann problem admits a Riemann solution as the constant state or the single rarefaction wave, it has already been proved by Matsumura–Nishihara that the solution to the Cauchy problem tends toward the constant state or the single rarefaction wave as the time goes to infinity. We investigate that the decay rate in time of the corresponding solutions and their derivative. These are the first results concerning the asymptotic decay of the solutions and their derivative to the Cauchy problem of the scalar conservation law with nonlinear viscosity. The proof is given by L^1, L^2-energy and time-weighted L^q-energy methods.

1. Introduction and main theorems

In this paper, we shall consider the asymptotic behavior of solutions for the one-dimensional scalar conservation law with a nonlinearly degenerate viscosity (p-Laplacian type viscosity with $p > 1$)

$$\begin{cases} \partial_t u + \partial_x \big(f(u) \big) = \mu \, \partial_x \left(|\, \partial_x u \,|^{p-1} \, \partial_x u \right) & (t > 0, x \in \mathbb{R}), \\ u(0,x) = u_0(x) & (x \in \mathbb{R}), \\ \lim\limits_{x \to \pm\infty} u(t,x) = u_\pm & \big(t \geq 0\big). \end{cases} \tag{1.1}$$

Here, $u = u(t,x)$ denotes the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity, $f = f(u)$ is the flux function depending only on u, μ is the viscosity coefficient, u_0 is the given initial data, and constants $u_\pm \in \mathbb{R}$ are the prescribed far field states. We suppose the given flux $f = f(u)$ is a C^3-function satisfying $f(0) = f'(0) = 0$, μ is a positive constant and far field states $u_\pm$ satisfy $u_- < u_+$ without loss of generality.

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At first, we shall motivate the physical meaning to the nonlinearly degenerate viscosity and review the related models concerning with the Cauchy problem (1.1). It is known that if $p = 1$ and $f(u) = \frac{1}{2}u^2$, the equation in our problem (1.1) becomes the viscous Burgers equation:

$$\partial_t u + u\,\partial_x u = \mu\,\partial_x^2 u.$$

In particular, the viscosity term $\mu\,\partial_x^2 u$ stands for Newtonian fluid. The Newtonian fluid is what satisfies the relation between the strain rate $\partial_{x_j} u_i + \partial_{x_i} u_j$ ($\partial_x u$, for one-dimensional case) is linear, that is,

$$\tau = \mu \left(\partial_{x_j} u_i + \partial_{x_i} u_j \right) \quad \text{or} \quad \tau = \mu\,\partial_x u.$$

On the other hand, if a fluid satisfies the relation between the strain rate and the stress is nonlinear (for example, polymers, viscoelastic or viscoplastic flow), the fluid is a non-Newtonian fluid, such as, blood, honey, butter, whipped cream, suspension, and so on. The typical nonlinearity in the non-Newtonian fluid is the power-law fluid (cf. [20]), that is,

$$\tau = \mu \left(\partial_{x_j} u_i + \partial_{x_i} u_j \right)^p \quad \text{or} \quad \tau = \mu \left(\partial_x u \right)^p.$$

Ladyženskaja [14] has proposed a new mathematical model for the incompressible Navier–Stokes equation with the power-law type nonlinear viscosity (see also [3]). The Ladyženskaja equation is the following:

$$\partial_t u_i + u_j\,\partial_{x_j} u_i = -\partial_{x_i} p + \partial_{x_j} \left(\left(\mu_0 + \mu_1 \left(\sum_{i,j} \left(\partial_{x_i} u_j \right)^2 \right)^{\frac{r}{2}} \right) \partial_{x_j} u_i \right) + f_i$$

where $i = 1, 2$, or $i = 1, 2, 3$. In particular, if $\mu_0 = 0, \mu_1 > 0$ and $r > -1$, this model is said to be the Ostwald–de Waele model:

$$\partial_t u_i + u_j\,\partial_{x_j} u_i = -\partial_{x_i} p + \partial_{x_j} \left(\left(\mu\,|\,D\vec{u}\,|^r \right) \partial_{x_j} u_i \right) + f_i$$

where $|\,D\vec{u}\,| := \left(\sum_{i,j} \left(\partial_{x_i} u_j \right)^2 \right)^{\frac{1}{2}}$, and $i = 1, 2$, or $i = 1, 2, 3$. In this sense, our viscosity $\mu\,\partial_x \big(|\,\partial_x u\,|^{p-1} \partial_x u \big)$ should be called the Ostwald–de Waele type viscosity.

We are interested in the asymptotic behavior and its precise estimates in time of the global solution to our problem (1.1). It can be expected that the large-time behavior is closely related to the weak solution ("Riemann solution") of the corresponding Riemann problem (cf. [16,33]) for the non-viscous hyperbolic part of (1.1):

$$\begin{cases} \partial_t u + \partial_x \big(f(u) \big) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0^{\mathrm{R}}(x) & (x \in \mathbb{R}), \end{cases} \tag{1.2}$$

where u_0^{R} is the Riemann data defined by

$$u_0^{\mathrm{R}}(x) = u_0^{\mathrm{R}}(x; u_-, u_+) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0). \end{cases}$$

In fact, for $p = 1$ in (1.1), the usual linear viscosity case:

$$\begin{cases} \partial_t u + \partial_x \big(f(u) \big) = \mu\,\partial_x^2 u & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) & (x \in \mathbb{R}), \\ \lim\limits_{x \to \pm\infty} u(t, x) = u_\pm & \big(t \geq 0 \big), \end{cases} \tag{1.3}$$

when the smooth flux function f is genuinely nonlinear on the whole space $\mathbb{R}$, i.e., $f''(u) \neq 0$ ($u \in \mathbb{R}$), Il'in–Oleĭnik [11] showed the following: if $f''(u) > 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single rarefaction wave solution, the global solution in time of the Cauchy problem (1.3) tends toward the rarefaction wave; if $f''(u) < 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single shock wave