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Persistence properties of the solutions to a generalized two-component Camassa–Holm shallow water system

Yuan Zhu, Fengyun Fu*

School of Mathematics and Statistics, Guangdong University of Finance and Economics, 510320 Guangzhou, PR China

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ABSTRACT

This paper is concerned with the Cauchy problem for a generalized two-component Camassa–Holm shallow water system. We prove that the solution will maintain the corresponding properties at infinity within its lifespan provided the initial data decay exponentially and algebraically, respectively.

1. Introduction

In this paper, we consider the following generalized two-component Camassa–Holm system (G2CH):

$$\begin{cases} m_t + \sigma(um_x + 2u_x m) + 3(1-\sigma)uu_x + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases}$$

where $m = u - u_{xx}$ and σ is a real parameter. (G2CH) was recently derived in [4] following Ivanov's modeling method [24]. It is a model from the shallow water theory with nonzero constant vorticity, where $u(t, x)$ is the horizontal velocity and $\rho(t, x)$ is related to the free surface elevation from equilibrium (or scalar density). The real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching. Recently, some mathematical properties

* Corresponding author.
E-mail addresses: zhuyuan7@mail2.sysu.edu.cn (Y. Zhu), fengyunfu@aliyun.com (F. Fu).

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such as wave-breaking phenomena, global existence of strong solution and stability of solitary waves of the Cauchy problem for (G2CH) have been studied in [4,5].

For $\rho \equiv 0$, (G2CH) becomes

$$u_t - u_{txx} + 3uu_x = \sigma(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

which models finite length, small amplitude radial deformation waves in cylindrical hyper-elastic rods [15]. Here $u(t,x)$ represents the radial stretch relative to a pre-stressed state.

For $\sigma = 1$, Eq. (1.1) becomes the celebrated Camassa–Holm equation (CH), modeling the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t,x)$ stands for the fluid velocity at time t in the spatial x direction [2,17]. CH is also a model for the propagation of axially symmetric waves in hyper-elastic rods [15]. It has a bi-Hamiltonian structure [19] and is completely integrable [2,8]. Also there is a geometric interpretation of CH in terms of geodesic flow on the diffeomorphism group of the circle [13]. Its solitary waves are peaked [3]. They are orbitally stable and interact like solitons [1,14]. The Cauchy problem for CH has been studied extensively [9–11,16,23]. It has been shown that this equation is locally well-posed [9,10,16] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Moreover, it has global strong solutions [9,10,7] and also finite time blow-up solutions [9–11,7]. The advantage of CH in comparison with the KdV equation lies in the fact that CH has peaked solitons and models wave breaking [3,11] (by wave breaking we understand that the wave remains bounded while its slope becomes unbounded in finite time [25]). In addition, persistence properties and unique continuation of the solution to CH have been studied in [23].

For $\rho \not\equiv 0$ and $\sigma = 1$, (G2CH) recovers the standard two-component Camassa–Holm system (2CH) which was recently derived rigorously in [24,12]. The Cauchy problems for (2CH) have been studied in many works, cf. [12,6,18,20–22]. Local well-posedness for (2CH) with the initial data in Sobolev spaces and in Besov spaces has been established in [12,18,21]. The blow-up phenomena and global existence of strong solutions to (2CH) in Sobolev spaces have been derived in [18,20–22].

Note that the following boundary assumption is required in the hydrodynamical derivation of (G2CH) [4], $u(t,x) \to 0$ and $\rho(t,x) \to 1$ as $|x| \to \infty$, at any instant t. Then, setting $\eta \triangleq \rho - 1$, we can rewrite (G2CH) as follows:

$$\begin{cases} m_t + \sigma u m_x + 2\sigma u_x m + 3(1-\sigma)uu_x + (\eta+1)\eta_x = 0, \\ \eta_t + u\eta_x + u_x(\eta+1) = 0. \end{cases} \tag{1.2}$$

Using the Green function $p(x) \triangleq \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ and the identity $(1-\partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$, we can set up the Cauchy problem for (G2CH):

$$\begin{cases} u_t + \sigma uu_x + \partial_x p * F(u,\eta) = 0, & t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + u_x(\eta+1) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \eta(0,x) = \eta_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $F(u,\eta) \triangleq \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\eta^2 + \eta$.

The goal of this paper is to investigate the persistence properties of the solution to System (1.3). We will prove that the solution maintains the corresponding properties at infinity within its lifespan provided the initial data decay exponentially and algebraically, respectively (see Theorems 3.1 and 3.2).

Our paper is organized as follows. In Section 2, we recall the local well-posedness of System (1.3) in Sobolev spaces and prove a useful lemma which is crucial to the proof of main theorems later. In Section 3, two persistence properties of the solution to System (1.3) are given.