Nonlinear Analysis 128 (2015) 86–99

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Characterizations of Sobolev spaces via averages on balls[☆]

Feng Dai[a], Amiran Gogatishvili[b], Dachun Yang[c,*], Wen Yuan[c]

[a] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada*
[b] *Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Prague 1, Czech Republic*
[c] *School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China*

ARTICLE INFO

Article history:
Received 23 February 2015
Accepted 24 July 2015
Communicated by S. Carl

MSC:
primary 46E35
secondary 42B25
42B35

Keywords:
Sobolev space
Average on ball
Difference
Euclidean space
Space of homogeneous type

ABSTRACT

In this paper, the authors prove several equivalent characterizations of Sobolev spaces of even integer orders on $\mathbb{R}^n$, using the average

$$B_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy$$

of a function f over the ball $B(x,t) := \{y \in \mathbb{R}^n : \ |y-x| < t\}$ with $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. These characterizations rely only on the metric and the Lebesgue measure on $\mathbb{R}^n$ and are simpler than those obtained recently by Alabern et al. (2012). Moreover, these results may shed new light on the theory of high order Sobolev spaces on spaces of homogeneous type.

1. Introduction

The problem of introducing Sobolev spaces on metric measure spaces where differential structures are not available is one of the central topics in analysis. A very important progress on this problem was achieved by Hajłasz [10], who successfully introduced a concept of gradients (now widely known as the Hajłasz gradients in literatures) and used it to introduce the first order Sobolev spaces on metric measure spaces. The Hajłasz

☆ This project is supported by the National Natural Science Foundation of China (Grant Nos. 11171027, 11361020, 11411130053 and 11471042), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2013YB60 and 2014KJJCA10). The work of the first author was supported in part by NSERC Canada under grant RGPIN 04702 Dai. The research of A. Gogatishvili was also partially supported by the grant P201/13/14743S of the Grant agency of the Czech Republic and RVO: 67985840.

* Corresponding author.
E-mail addresses: fdai@ualberta.ca (F. Dai), gogatish@math.cas.cz (A. Gogatishvili), dcyang@bnu.edu.cn (D. Yang), wenyuan@bnu.edu.cn (W. Yuan).

http://dx.doi.org/10.1016/j.na.2015.07.024
0362-546X/

gradients have become a powerful tool in the study of the first order Sobolev spaces on metric measure spaces; see, for example, [10–12,16,25]. After the pioneering work of Hajłasz [10], several different approaches were proposed by different authors to introduce and study first-order Sobolev spaces on metric measure spaces (see, for example, [8,18,12,11,25,15]). Indeed, great success has been achieved on the theory of the first order Sobolev spaces on metric measure spaces over the last two decades. On the other hand, however, the problem of developing a successful theory of higher order Sobolev spaces on metric measure spaces remains open and has attracted a lot of attentions in recent years.

Very recently, Alabern, Mateu and Verdera [1] obtained an interesting new characterization of Sobolev spaces on $\mathbb{R}^n$, which relies only on the metric and the Lebesgue measure on $\mathbb{R}^n$ and hence provides a possible way to introduce Sobolev spaces of arbitrary order of smoothness on any metric measure space. To describe this new characterization, we first recall that the (inhomogeneous) Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ on $\mathbb{R}^n$ consist of all functions f on $\mathbb{R}^n$ such that $\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^n)} < \infty$. Here and throughout this article, the smoothness index α is any positive real number, $p \in (1, \infty)$, $\Delta := \sum_{i=1}^n (\frac{\partial}{\partial x_i})^2$ is the Laplacian, and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian defined in terms of the distributional Fourier transform via $((-\Delta)^{\alpha/2} f)^\wedge(\xi) := |\xi|^\alpha \widehat{f}(\xi)$ for any tempered distribution f. Next, we recall a well-known classical characterization of $W^{\alpha,p}(\mathbb{R}^n)$ via square functions (see, for example, [24,19,20,25]), which asserts that, for $\alpha \in (0, 1)$ and $p \in (1, \infty)$, $f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $s_\alpha(f) \in L^p(\mathbb{R}^n)$, where $s_\alpha(f)$ is the square function given by

$$s_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left[⨍_{B(\cdot,\, t)} |f(\cdot) - f(y)|\, dy \right]^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}. \tag{1.1}$$

Here and hereafter, we use the following notation: for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$B(x,t) := \{y \in \mathbb{R}^n : \ |y - x| < t\},$$
$$⨍_{B(x,t)} g(y)\, dy := \frac{1}{|B(x,t)|} \int_{B(x,t)} g(y)\, dy =: B_t g(x). \tag{1.2}$$

Such a characterization, however, fails for $\alpha \geq 1$. Indeed, it is known that, if $\alpha \geq 1$, then $\|f\|_{L^p(\mathbb{R}^n)} + \|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty$ implies $f \equiv 0$ on $\mathbb{R}^n$ (see [9, Section 4]).

In order to have a similar characterization for $W^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \geq 1$, Alabern, Mateu and Verdera [1] introduced a new square function S_α, with a slight modification of the definition of $s_\alpha(f)$ via dropping the absolute value in $|f(\cdot) - f(y)|$ of (1.1), given by

$$S_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left| ⨍_{B(\cdot,\, t)} [f(\cdot) - f(y)]\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{1.3}$$

It turns out that such a modification is significant enough for the authors of [1] to establish a characterization for all Sobolev spaces of smoothness orders $\alpha \in (0, 2)$: for $\alpha \in (0, 2)$ and $p \in (1, \infty)$, $f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $S_\alpha(f) \in L^p(\mathbb{R}^n)$. The key point here is that, unlike the classical square function s_α in (1.1), this new function S_α in (1.3) provides smoothness up to order 2, namely, for $f \in C^2(\mathbb{R}^n)$ and $t \in (0,\, 1)$,

$$⨍_{B(x,\, t)} [f(x) - f(y)]\, dy = O(t^2), \quad x \in \mathbb{R}^n. \tag{1.4}$$

This phenomenon, followed directly from the Taylor expansion, was first observed by Wheeden in [23] (see also [24]) and later independently by Alabern, Mateu and Verdera in [1].

A more complicated characterization of $W^{\alpha,p}(\mathbb{R}^n)$ for higher orders of smoothness (i.e., $\alpha \geq 2$) was also established in [1, Theorems 2 and 3]. To be precise, assume that $\alpha \in [2N, 2N + 2)$ with $N \in \mathbb{N} := \{1, 2, \ldots\}$.