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Forced oscillations of a massive point on a compact surface with a boundary

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ABSTRACT

We present sufficient conditions for the existence of a periodic solution for a class of systems describing the periodically forced motion of a massive point on a compact surface with a boundary.

1. Brief introduction

In 1922, G. Hamel proved [4] that equations describing the motion of a periodically forced pendulum have at least one periodic solution. Since then, many results concerning periodic solutions in pendulum-like systems have been obtained by various authors including theorems for a one-dimensional forced pendulum [5], a result by M. Furi and M. P. Pera [3], who showed that a frictionless spherical pendulum also has forced oscillations, and work by V. Benci and M. Degiovanni [1], who studied the motion of a massive point on a compact boundaryless surface with friction and presented sufficient conditions for the existence of forced oscillations. As far as we know, the case of a compact surface with a boundary is far less developed.

However, surfaces with boundaries naturally appear in various mechanical systems. For instance, in a book [2] by R. Courant and H. Robbins, the authors consider the system of an inverted planar pendulum placed on the floor of a train carriage, and show that for any law of motion for the train, there always exists at least one initial position such that the pendulum, starting its motion from this position with zero generalized velocity, moves without falling for an arbitrary long time. Here, the compact surface is a half-circle and its boundary is the two-pointed set.

Topological ideas, which lie in the basis of the above result, can be rigorously justified [6] – in [2] some details are omitted – and generalized for different types of systems. Moreover, it was proved [6] that for an inverted pendulum with a periodic law of motion for its pivot point, there exists a periodic solution along

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which the pendulum never becomes horizontal, i.e. it never falls. This was obtained as an application of a topological theorem by R. Srzednicki, K. Wójcik, and P. Zgliczyński [7].

In the current paper we further develop this result [6] and present sufficient conditions for the existence of a periodic solution for a class of systems describing periodically forced motion with friction of a massive point on a compact surface with a boundary and non-zero Euler–Poincaré characteristic. We prove that if for the considered system all solutions that are tangent to the boundary are externally tangent to it, then there exists at least one periodic solution that never reaches the boundary.

2. Main result

2.1. Governing equations

In this subsection, we introduce governing equations for a mechanical system consisting of a massive point moving with friction-like interaction on a surface and prove a lemma, which we are going to use further in our main theorem. For the sake of simplicity, we assume that all manifolds and considered functions are smooth (i.e. C^∞).

Let M be a compact connected two-dimensional manifold with a boundary embedded in $\mathbb{R}^3$. Manifold M describes the surface on which a massive point moves. Its boundary is a finite collection of curves, which are homeomorphic to circles. We also assume that the point moves with friction, which we will specify further below.

In our further consideration, we will study the behavior of our system in a vicinity of ∂M. In this regard, it is convenient to consider an enlarged manifold M^+. Let M^+ be a boundaryless connected two-dimensional manifold also embedded in $\mathbb{R}^3$ such that $M \subset M^+$. Therefore, the motion of the massive point can be described by a function of time $q\colon \mathbb{R} \to M^+$. Note that there are infinitely many possibilities for constructing M^+ but for our use they are all the same.

In general form, the equations of motion can be written as follows:

$$m\ddot{q} = F + F_{friction} + F_{constraint}.$$

Here m is the mass of the point; $F\colon \mathbb{R}/T\mathbb{Z} \times TM^+ \to \mathbb{R}^3$ is a T-periodic force acting on the point; $F_{friction}\colon \mathbb{R}/T\mathbb{Z} \times TM^+ \to \mathbb{R}^3$ is a friction-like force which, for a given t, q and $\dot{q}$, we assume to have the following form:

$$F_{friction} = -\dot{q}\gamma(t, q, \dot{q}), \quad \gamma\colon \mathbb{R}/T\mathbb{Z} \times TM^+ \to \mathbb{R}.$$

The force of constraint has usual form $F_{constraint} = \lambda n_q$, where $\lambda \in \mathbb{R}$ and n_q is a normal vector to M^+ at point q.

Finally, assuming without loss of generality that $m = 1$, we obtain the following equations of motion:

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= F(t, q, p) - p\gamma(t, q, p) + \lambda n_q. \end{aligned} \tag{1}$$

Note that one can get rid of unknown parameter λ in (1) in the usual way by projecting the right-hand sides of the above equations to $T_q M^+$.

Lemma 2.1. *Suppose that there exists a constant $d > 0$ such that in (1)*

$$\inf_{\substack{t\in[0,T],\ q\in M \\ \|p\|>d}} \gamma(t, q, p) > 0, \tag{2}$$