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On the energy inequality for weak solutions to the Navier–Stokes equations of compressible fluids on unbounded domains

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ABSTRACT

We consider the Navier–Stokes equations of compressible isentropic viscous fluids on an unbounded three-dimensional domain with a compact Lipschitz boundary. Under the condition that the total mass of the fluid is finite, we show the existence of globally defined weak solutions satisfying the energy inequality in differential form.

1. Introduction

Let $T > 0$ be fixed and let $\Omega \subset \mathbb{R}^3$ be an exterior domain (an unbounded domain with compact Lipschitz boundary $\partial\Omega$). We consider the Navier–Stokes equations of a compressible isentropic viscous fluid

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho^\gamma = \mathrm{div}_x \mathbb{S}, \tag{1.2}$$

in the unknown variables

$$\varrho = \varrho(x,t) : \Omega \times (0,T) \to \mathbb{R} \quad \text{and} \quad \mathbf{u} = \mathbf{u}(x,t) : \Omega \times (0,T) \to \mathbb{R}^3$$

representing the density and velocity of the fluid, respectively. The adiabatic constant γ is subjected to the technical constraint

$$\gamma > \frac{3}{2},$$

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while the viscous stress tensor $\mathbb{S}$ fulfills *Newton's rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},$$

with shear viscosity coefficient μ and bulk viscosity coefficient η satisfying

$$\mu > 0 \quad \text{and} \quad \eta \geq 0.$$

Accordingly, we may write

$$\mathrm{div}_x \mathbb{S} = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla_x \mathrm{div}_x \mathbf{u},$$

where

$$\lambda = \eta - \frac{2}{3}\mu.$$

As the underlying physical domain is unbounded, the system is supplemented with the far field conditions

$$\lim_{|x|\to\infty} \varrho(x,t) = 0, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0,$$

and the no-slip boundary condition

$$\mathbf{u}(x,t)_{|x\in\partial\Omega} = 0.$$

Finally, we prescribe the initial conditions

$$\varrho(0) = \varrho_0 \quad \text{and} \quad (\varrho\mathbf{u})(0) = \boldsymbol{q}_0$$

where ϱ_0 and $\boldsymbol{q}_0$ are given functions, complying with the following assumptions:

- $\varrho_0 \in L^1(\Omega) \cap L^\gamma(\Omega)$ and $\varrho_0 \geq 0$ almost everywhere.
- $\boldsymbol{q}_0 \in L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}^3)$ is such that $\boldsymbol{q}_0(x) = 0$ whenever $\varrho_0(x) = 0$. Moreover

$$\frac{|\boldsymbol{q}_0|^2}{\varrho_0} \in L^1(\Omega).$$

Under the above general assumptions on the structural coefficients and the initial data, the compressible Navier–Stokes equations (1.1)–(1.2) are known to admit at least one globally defined weak solution[1] $(\varrho, \mathbf{u})$ (see [5,12,14]). In particular, for the energy functional

$$E(t) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2(t) + \frac{1}{\gamma-1}\varrho^\gamma(t)\right] \mathrm{d}x,$$

the weak solutions can be constructed to fulfill the *energy inequality in the integral form*

$$E(t) + \int_0^t \int_\Omega \left[\mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2\right] \mathrm{d}x\,\mathrm{d}r \leq E_0 \tag{1.3}$$

for almost any $t \in [0,T]$, where

$$E_0 = \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{q}_0|^2}{\varrho_0} + \frac{1}{\gamma-1}\varrho_0^\gamma\right] \mathrm{d}x.$$

[1] See Section 3 for the precise definition.