Nonlinear Analysis 128 (2015) 325–347

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Boundedness of the maximal operator and Sobolev's inequality on non-homogeneous central Herz–Morrey–Orlicz spaces

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A R T I C L E I N F O

Article history:
Received 23 April 2015
Accepted 18 August 2015
Communicated by Enzo Mitidieri

MSC:
primary 31B15
46E35

Keywords:
Maximal operator
Non-homogeneous
Central Herz–Morrey–Orlicz spaces
Riesz potentials
Sobolev's inequality

A B S T R A C T

Our aim in this paper is to deal with the boundedness of the Hardy–Littlewood maximal operator in non-homogeneous central Herz–Morrey–Orlicz spaces. As an application, we give Sobolev's inequality for Riesz potentials.

1. Introduction

Let $\mathbf{R}^N$ be the Euclidean space. In [3], Beurling introduced the space $B^p(\mathbf{R}^N)$ to extend Wiener's ideas [23,24] which describes the behavior of functions at infinity. Feichtinger [8] gave an equivalent norm on $B^p(\mathbf{R}^N)$, which is a special case of norms in Herz spaces $K_p^{\alpha,r}(\mathbf{R}^N)$ introduced by Herz [13]. More precisely, $B^p(\mathbf{R}^N) = K_p^{-N/p,\infty}(\mathbf{R}^N)$ (see also [11]). Alvarez, Guzmán-Partida and Lakey [2] defined the central Morrey spaces $B^{p,\lambda}(\mathbf{R}^N)$ to study the relationship with λ-central bounded mean oscillation spaces, where $B^{p,0}(\mathbf{R}^N) = B^p(\mathbf{R}^N)$.

In [10], García-Cuerva studied the boundedness of the maximal operator on the space $B^p(\mathbf{R}^N)$. Further, Li and Yang [14] showed that the maximal operator is bounded on homogeneous Herz spaces and non-homogeneous Herz spaces. Our first aim in this paper is to introduce non-homogeneous central

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http://dx.doi.org/10.1016/j.na.2015.08.014
0362-546X/

Herz–Morrey–Orlicz spaces $\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ as an extension of $K_p^{\alpha,r}(\mathbf{R}^N)$, and study the boundedness of the Hardy–Littlewood maximal operator (see Theorem 3.9).

In classical Lebesgue spaces, we know Sobolev's inequality:

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^N)} \le C\|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N), 0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see, e.g. [1, Theorem 3.1.4]). Fu, Lin and Lu [9] showed Sobolev's inequality for $B^{p,\lambda}(\mathbf{R}^N)$ (see also [14] for non-homogeneous Herz spaces, [4,5] for non-homogeneous Herz–Morrey spaces, and [16] for non-homogeneous central Morrey spaces). Our second aim is to give Sobolev's inequality for Riesz potentials of functions in non-homogeneous central Herz–Morrey–Orlicz spaces (see Theorem 4.5).

Suppose $f \in \mathcal{H}^{p,q,\omega}(\mathbf{R}^N)$, that is, it satisfies an L^p integrability such as

$$\int_1^\infty \left\{\omega(r)\|f\|_{L^p(A(0,r))}\right\}^q \frac{dr}{r} < \infty \quad \text{when } 0 < q < \infty,$$
$$\sup_{r>1} \omega(r)\|f\|_{L^p(A(0,r))} < \infty \quad \text{when } q = \infty,$$

where ω is a doubling weight, $1 < p < \infty$ and $A(0,r) = B(0,2r) \setminus B(0,r)$ is the annulus with $B(x,r)$ denoting the open ball centered at x of radius r. Then we want to find p_1 and a weight τ such that $I_\alpha f \in \mathcal{H}^{p_1,q,\tau}(\mathbf{R}^N)$ (see Theorems 4.5 and 4.11 and their remarks). In the borderline case $\alpha p = N$, instead of Trudinger's inequality, we show the weighted L^p integrability

$$\int_{\mathbf{R}^N} \left\{(1+|x|)^{-N/p}(\log(e+|x|))^{-1+\theta}|I_\alpha f(x)|\right\}^p dx \le C \int_{\mathbf{R}^N} \left\{(\log(e+|y|))^{\theta}|f(y)|\right\}^p dy$$

as in Edmunds and Triebel [7]; see Theorem 4.12.

Since it may happen that $I_\alpha|f| \equiv \infty$ for some $f \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$, we modify the Riesz kernel I_α by

$$I_{\alpha,k}(x,y) = \begin{cases} I_\alpha(x-y) & \text{when } |y| < 1, \\ I_\alpha(x-y) - \displaystyle\sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) & \text{when } |y| \ge 1 \end{cases}$$

for a nonnegative integer k (see [17,18]); $I_{\alpha,0}$ is the usual Riesz kernel I_α of order α. Then our third task is to find k such that the generalized Riesz potential

$$I_{\alpha,k}f(x) = \int_{\mathbf{R}^N} I_{\alpha,k}(x,y)f(y)\,dy$$

is well defined for almost every $x \in \mathbf{R}^N$ and belongs to a suitable non-homogeneous central Herz–Morrey–Orlicz space (see Theorem 5.4).

Finally, following Gogatishvili–Mustafayev [12], we study the duality properties between $\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ and $\overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ (for the definition of $\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ and $\overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$, see Section 2).

2. Preliminaries

Let us consider a function

$$\Phi(t) = t\phi(t) : [0,\infty) \to [0,\infty)$$

with ϕ satisfying the following conditions: