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Convexity of the images of small balls through nonconvex multifunctions

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This paper is dedicated to Boris Teodorovich Polyak, on the occasion of his 80th birthday

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ABSTRACT

In the present paper, the following convexity principle is proved: any closed convex multifunction, which is metrically regular in a certain uniform sense near a given point, carries small balls centred at that point to convex sets, even if it is perturbed by adding $C^{1,1}$ smooth mappings with controlled Lipschitzian behaviour. This result, which is valid for mappings defined on a subclass of uniformly convex Banach spaces, can be regarded as a set-valued generalization of the Polyak convexity principle. The latter, indeed, can be derived as a special case of the former. Such an extension of that principle enables one to build large classes of nonconvex multifunctions preserving the convexity of small balls. Some applications of this phenomenon to the theory of set-valued optimization are then proposed and discussed.

1. Introduction

Treating problems from mathematical programming, optimal control and from several areas of mathematical economics yields a tremendous demand of convexity. Convexity assumptions on problem data often strengthen the analysis tools and trigger the application of special approaches, otherwise not practicable. Even though such a demand has led to deepen our knowledge about convexity and then to develop expanding branches of convex analysis, many fundamental issues about convexity still remain to be investigated. In the author's opinion, one of such issues concerns the behaviour of convex sets under nonlinear transformations. Indeed, not much seems to be known so far about those sets whose image through nonlinear mappings is convex. The existing results on this question can be schematically classified as "around a point" (local) results or

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as “on set” (nonlocal) results. As an example of nonlocal result the Lyapunov convexity theorem on the range of a vector measure occupies a prominent place (see [14]). It found notable applications in control theory and mathematical economics (see [1,17]). Other examples of global results are, for instance, those in [5,20,21]. As an example of local result, the Polyak convexity principle is certainly to be mentioned (see [18,19]). Like the Lyapunov’s theorem, it revealed to be useful in several topics of optimization and control theory, by providing conditions upon which nonlinear mappings carry small balls around a point to convex sets.

The present paper aims at bringing some contributions in the same vein as the Polyak convexity principle, but entering now the realm of set-valued mappings. The starting point of the analysis here proposed is the well-known fact that convex multifunctions (i.e. set-valued mappings with convex graph) carry any convex set to a convex set. If considering the category, whose objects are convex sets, this class of mappings seem to naturally play the role of category morphisms. Unfortunately, by simple examples it is readily realized that, when adding a nonlinear single-valued mapping to a convex multifunction, in general the convex graph property of the latter is destroyed. Thus the question arises under which conditions mappings, obtained by perturbing convex multifunctions by nonlinear mappings, still carry small balls to convex sets. The main result of this paper provides an answer to this problem. It states that, if to a convex multifunction, which is metrically regular near a reference point uniformly over its image, a $\mathrm{C}^{1,1}$ mapping is added, whose Lipschitzian behaviour is controlled by the modulus of regularity of the former, then the resulting set-valued mapping preserves the convexity of small balls around the reference point of its domain. In fact, this result can be regarded as an extension of the Polyak convexity principle to a large class of set-valued mappings. As it happens for its single-valued counterpart, it is valid for mappings defined on uniformly convex Banach spaces having second order polynomial modulus of convexity. This class of spaces includes, for instance, all Hilbert spaces. The proof combines a nice property, coming from the rotund geometry of balls in the aforementioned class of Banach spaces, with a convex solvability behaviour of set-valued mappings, that are perturbed as described. The latter is a consequence of the persistence of metric regularity under additive Lipschitz perturbations, a well-known phenomenon in variational analysis, which has revealed to be useful in various contexts related to the solution stability and sensitivity for generalized equations (see [9,16]).

The contents of the paper are organized as follows. In Section 2 some tools, mainly from geometric functional analysis and from nonlinear analysis, that are needed for establishing the main result, are recalled. In particular, in Section 2.3 a strengthened notion of metric regularity for set-valued mappings is introduced. Several classes of multifunctions satisfying such a special property are exhibited, while it is observed that the original notion of metric regularity is weaker (in the sense that it holds more generally). In Section 3 the main result is proved and commented. Then, it is shown how from this wider convexity principle the Polyak’s one can be derived, as a special case. To illustrate the main result, a specialization of it to a wide class of multifunctions and particular examples are also discussed. Section 4 is reserved to illustrate an application of the main result to a topic from set-valued optimization. More precisely, a class of optimization problems is considered, whose set-valued objective is expressed as a sum of a single-valued and a set-valued mapping. This structure in the objective mapping may model noise effects on vector optimization problems. In this context, the convexity principle, under certain additional assumptions, leads first of all to establish the existence of efficient pairs for certain localizations of an unconstrained problem, and then to achieve optimality conditions based on the Lagrangian scalarization.

2. Tools from nonlinear analysis

2.1. Uniformly convex Banach spaces

The analysis of the posed problem will be carried out in the particular setting of the uniformly convex real Banach spaces. This is because the main result presented in the paper essentially relies on certain