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Nonexistence and existence results of an optimal control problem governed by a class of multisolution semilinear elliptic equations ☆

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ABSTRACT

An optimal control problem governed by a class of multisolution semilinear elliptic equations is studied in this paper. It is pointed out that the control domain and cost functional are nonconvex. By analyzing the maximum condition for an optimal relaxed control, one nonexistence and some existence results of an optimal pair are obtained. The idea of using relaxed controls is mainly derived from the paper (Lou, 2007) by Lou where a linear and well-posed controlled system was considered. The present paper is mainly devoted to discussing the case not contained in Lou (2007).

1. Introduction

Existence of optimal controls is an important topic in the theory of optimal controls and it has some actual sense in the practice. By summarizing the existing literatures, it can be found that to obtain an existence result an assumption of convexity on a control domain and cost functional is very important. In general, if the convexity is assumed an existence result can be obtained by a method of minimizing sequence (see [12,14]). In the absence of convexity, the problem will be more difficult. To the best of our knowledge, the first result on the existence of optimal controls in the absence of convexity was established by Neustadt [19] for the finite-dimensional linear systems. Later, more general cases were studied in the literatures (see [1,3–6,21]). However, for the infinite-dimensional case, relevant results are not rich enough. The readers can refer to

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[2,8,9,11,20] and the references cited therein. When a convex condition is not satisfied, relaxed controls have been proved to be an important tool for studying existence of optimal controls (see [15–17] for example). In [15], the author discussed existence and nonexistence of optimal controls for linear systems by using relaxed controls and the idea will be introduced later in detail. Motivated by [15], the semilinear case was considered in [17]. Nevertheless, all the controlled systems mentioned above are well-posed. For the case of non-well-posed controlled system, to the best of our knowledge, little is known about an existence result.

In the present paper, we discuss an optimal control problem governed by a class of semilinear elliptic equations. It is firstly mentioned that the state equation has exactly two solutions with respective to one control and hence is non-well-posed. In the theory of partial differential equations, non-well-posed equations are very important, which can describe many actual problems. As mentioned by Crandall and Rabinowitz [7] and Lions [13], non-well-posed equations can be used to describe bifurcation phenomena, enzymatic reactions, phenomena in plasma physics and chemistry and so on. On the other hand, the control domain and cost functional we considered are nonconvex. Our purpose is to obtain nonexistence and existence results of an optimal pair under the different cases. However, the main goal is to consider one case not given in [15]. The method used in this paper is similar to that is in [15]. The main idea can be stated as follows. Firstly the corresponding relaxed control problem for the initial control problem is introduced. Secondly it is devoted to deriving the maximum principle for an optimal relaxed pairs. Compared with [15–17], in our case this step is more complex and difficult since the relaxed controlled system is also non-well-posed. Finally the nonexistence or existence results of the initial problem can be obtained by proving whether an optimal relaxed control is an optimal control. In the final step, the main work is to analyze the maximum principle for an optimal relaxed pair and the adjoint system detailedly. We need to deduce the sign of a solution of the state equation and adjoint equation. Thus, unavoidably, the multisolution of state equation will bring about some difficulties and this is exactly the reason that the special controlled system is considered in this paper. That is to say that it is difficult to generalize our result to a more general non-well-posed controlled system. Nevertheless, it is worth mentioning that we obtain a nonexistence result. Checking the existing literatures on existence of optimal controls, one can find that negative results appear rarely.

The rest of this paper is organized as follows In Section 2, we give the formulation of the optimal control problem. Section 3 is devoted to presenting some preliminary results for a solution of the state equation. The Pontryagin's maximum principle for an optimal relaxed pair is obtained in Section 4. Finally we will state and prove the main results in Section 5.

2. Formulation of the optimal control problem

Let Ω be a bounded domain of $\mathbb{R}^N (N \leq 4)$ with a smooth boundary Γ. Consider the following elliptic controlled system:

$$\begin{cases} -\triangle y = \lambda y^3 + u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\lambda > 0$ is a fixed constant, u is a control and y is a state. Let the control domain $U = \{a, b\}$ and the set of admissible controls be defined by

$$\mathcal{U} \equiv \{u : \Omega \to U | u \text{ is measurable}\}.$$

First, for any $u \in \mathcal{U}$, we give the definition of a solution of (2.1).

Definition 2.1. A function $y \in H_0^1(\Omega)$ is called a solution of (2.1), if it satisfies

$$\int_\Omega \nabla y \cdot \nabla \varphi dx = \lambda \int_\Omega y^3 \varphi dx + \int_\Omega u \varphi dx$$

for any $\varphi \in H_0^1(\Omega)$.