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Elliptic equations with general singular lower order term and measure data

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ABSTRACT

In this paper we study a nonlinear elliptic boundary value problem with a general singular lower order term, whose model is

$$\begin{cases} -\Delta u = H(u)\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{on } \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, μ is a nonnegative bounded Radon measure on Ω and H is a continuous positive function outside the origin such that $\lim_{s\to 0^+} H(s) = +\infty$. We do not require any monotonicity property on the singular function H.

1. Introduction

We are interested in the existence of a distributional solution for the following quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = H(u)\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a nonnegative bounded Radon measure on an open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 2)$, H is a continuous positive function outside the origin such that $\lim_{s\to 0^+} H(s) = +\infty$ and $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^2, \quad \alpha > 0, \tag{1.2}$$

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$$|a(x,\xi)| \le \beta|\xi|, \quad \beta > 0, \tag{1.3}$$

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') > 0, \tag{1.4}$$

for every ξ, ξ' in $\mathbb{R}^N (\xi \neq \xi')$ and for almost every x in Ω.

Problem (1.1) is singular since one asks the solution to be zero on the boundary of Ω and so, H being neither continuous nor bounded in the origin, one of the main tasks will be to give a weak sense to the right hand side of (1.1).

Problem (1.1) has been extensively studied in the past under different assumptions on the function H and on the measure μ.

For the non singular case with $H \equiv 1$ we recall, among others, [1,3,8,19]. Moreover, in [15], Murat and Porretta studied the general non singular case with H a real, continuous and bounded function, proving the existence of a renormalized solution of (1.1).

The singular and monotone case $H(u) = \frac{1}{u^\gamma}$, with $\gamma > 0$, has been approached by different authors. First of all we recall the pioneering works by Crandall, Rabinowitz and Tartar [7] and by Lazer and McKenna [14], in which the case of Laplacian operator and smooth data μ and Ω has been studied. More recently, in [6], Boccardo and Orsina studied the uniformly elliptic case with datum $\mu = f \in L^1(\Omega)$. This work has been generalized to the case of the p-Laplacian operator in [9], to the case with nonhomogeneous right hand side in [16] and to the case with diffuse measure data in [17].

The aim of the present work is to extend [15] to the singular case of an unbounded H without using any monotonicity methods to handle the right hand side.

The focal point in the paper by Boccardo and Orsina was the possibility to construct a monotone approximating sequence of solutions provided with a local uniform positivity property on compact subsets of Ω; this allows to solve the singular problem in distributional sense.

In our framework this monotonicity property is lost because of two different reasons. On one hand we deal with general nonnegative Radon measure data, that cannot be approximated in a monotone way. On the other hand, H may not be a decreasing function. In the case in which the diffuse part of the datum is not identically zero, we will overcome these difficulties through a comparison argument that uses the renormalized formulation of some approximating problems.

We refer the reader to [11,12] for another possible approach to the singular problem (1.1) in the non-monotone case with linear differential operator and data $\mu = f \in L^r(\Omega), r > \frac{N}{2}$.

The plan of the paper is as follows. In Section 2 we will recall some definitions and basic results that we will often use in the sequel. Then, using that the Radon measure μ can be uniquely decomposed as $\mu_d + \mu_c$, where μ_d is diffuse with respect to the harmonic capacity and μ_c is concentrated on a set of zero harmonic capacity (see Section 2), in Section 3 we will give an approximation of problem (1.1) when the diffuse part μ_d of μ is not identically zero and we will prove a priori estimates on the sequence of approximating solutions. In Section 4, we will prove the existence of a suitable solution of (1.1) in the case $\mu_d \not\equiv 0$ and, finally, in Section 5, we will analyze the case of a purely singular measure datum, namely the case in which diffuse part μ_d is identically zero.

2. Notations and preliminaries

In the following Ω will be an open bounded subset of $\mathbb{R}^N (N \ge 2)$.

We denote by $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, so that $C(\mathbb{R}^*)$ is the space of functions on $\mathbb{R}^*$ which are continuous outside the origin, and by $C_b(\mathbb{R})$ the space of continuous and bounded functions on $\mathbb{R}$.

If not otherwise specified, we will denote by C several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance C can depend on Ω, γ, N) but they will never depend on the indexes of the sequences we will introduce. Moreover, in