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On the normal exponential map in singular conformal metrics

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ABSTRACT

Brake orbits and homoclinics of autonomous dynamical systems correspond, via Maupertuis principle, to geodesics in Riemannian manifolds endowed with a metric which is singular on the boundary (Jacobi metric). Motivated by the classical, yet still intriguing in many aspects, problem of establishing multiplicity results for brake orbits and homoclinics, as done in Giambò et al. (2005, 2010, 2011), and by the development of a Morse theory in Giambò et al. (2014) for geodesics in such kind of metric, in this paper we study the related normal exponential map from a global perspective.

1. Introduction

The purpose of this paper is to prove global regularity results for the distance-to-the-boundary function in Riemannian manifolds with singular metrics on the boundary. This kind of study is motivated by the use of the degenerate Jacobi metric (via Maupertuis' principle) for the problem of brake orbits and homoclinics in the autonomous case, as done in [5,6,8,7]. This approach was suggested for the first time by Seifert in [18], where a famous conjecture concerning a multiplicity results for brake orbits was formulated. The metric singularity on the boundary is of a very special type, being produced by the first order vanishing of a conformal factor which multiplies a fixed background metric. Following the theory developed in [9], in this paper we will introduce a suitable notion of normal exponential map adapted to this type of degenerate boundaries, and we will determine its regularity properties.

There exists a huge amount of literature concerning the study of brake orbits – see e.g., [10,13,14,19,20] – and more generally on the study of periodic solutions of autonomous Hamiltonian systems with prescribed energy [11,12,15–17]. We also observe here that manifolds with singular boundary of the type investigated

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in the present paper arise naturally in the study of certain compactifications of incomplete Riemannian manifolds. They constitute an important class of the so-called *singular manifolds*, see [1] and the references therein, where the singularity is described by the vanishing (or the diverging) of some conformal factor, called the *singularity function*.

In order to describe the results of the present paper, let us consider a Riemannian manifold (M, g) of class C^3, representing the configuration space of some conservative dynamical system, and let $V : M \to \mathbb{R}$ be a map of class C^2 on M, which represents the potential function of the system. Fix an energy level $E \in \mathbb{R}$, $E > \inf_M V$, and consider the Jacobi metric:

$$g_* = \tfrac{1}{2}(E - V)g, \tag{1.1}$$

defined in the open sublevel $V^{-1}\big(]-\infty, E[\big)$, the so called *potential well*. Note that g_* is singular on the boundary $V^{-1}(E)$.

For any $Q \in V^{-1}\big(]-\infty, E[\big)$, denote by $d_V(Q)$ the distance of Q from $V^{-1}(E)$ with respect to the Jacobi metric (1.1). In the recent work [9], the following assumptions:

- V is of class C^2 in a neighborhood of $V^{-1}\,(]-\infty, E[)$;
- E is a regular value for V;
- the sublevel $V^{-1}\,(]-\infty, E])$ is compact;

were used to prove that if the minimizer that realizes $d_V(Q)$ is unique, then d_V is differentiable at Q, and its gradient with respect to the Riemann metric g is given by

$$\nabla^g d_V(Q) = \frac{E - V(Q)}{2d_V(Q)}\dot\gamma_Q(1),$$

where γ_Q is the minimizer (affinely parametrized in the interval $[0, 1]$) joining $V^{-1}(E)$ with Q.

Note that this result has a global nature, and it provides a starting motivation to define and study Jacobi fields along Jacobi geodesics starting from $V^{-1}(E)$. This was done in [9, Section 4] where the Morse Index Theorem was proved.

Uniqueness of the minimizer is guaranteed for all points Q sufficiently close to the boundary $V^{-1}(E)$, and this case was first studied in [18] and later in [2,3].

Following the study in [9], in this paper we introduce a normal exponential map $\exp^\perp$, defined in terms of g_*-geodesics $\gamma :]0, a] \to V^{-1}\big(]-\infty, E[\big)$ satisfying $\lim_{s\downarrow 0}\gamma = P \in V^{-1}(E)$, see Section 2. Such geodesic is necessarily "orthogonal" to $V^{-1}(E)$, in the sense that a suitable normalization of $\dot\gamma(s)$, when s goes to 0, admits as limit as a vector $v \in T_P\big(V^{-1}(E)\big)$ which is g-orthogonal to $V^{-1}(E)$ at P. We prove the regularity of $\exp^\perp$, and we establish the equivalence between conjugate points to $V^{-1}(E)$ and critical values of the exponential map (Proposition 2.3 and Theorem 2.9).

In Section 3 we apply the above result to prove that if the minimizer between $V^{-1}(E)$ and Q_0 is unique and if Q_0 is not conjugate to $V^{-1}(E)$, then $d_V(Q)$ is of class C^2 in a neighborhood of Q_0 (cf. Theorem 3.1).

2. Exponential map and focal points

A geodesic $x : I \subset \mathbb{R} \to V^{-1}\big(]-\infty, E[\big)$ relative to the metric g_* (1.1) will be called a *Jacobi geodesic*; such a curve satisfies the second order differential equation:

$$\big(E - V(x(s))\big)\tfrac{\mathrm{D}}{\mathrm{d}s}\dot x(s) - g\big(\nabla V(x(s)), \dot x(s)\big)\dot x(s) + \frac{1}{2}g\big(\dot x(s), \dot x(s)\big)\nabla V(x(s)) = 0. \tag{2.1}$$