Nonlinear Analysis 127 (2015) 55–70

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

# An extension of Dragilev's theorem for the existence of periodic solutions of the Liénard equation

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ARTICLE INFO

*Article history:*
Received 27 April 2015
Accepted 26 June 2015
Communicated by Enzo Mitidieri

*MSC:*
34C05
34C25
34C15

*Keywords:*
Liénard equation
Limit cycles
Dragilev's theorem

ABSTRACT

The problem of the existence of periodic solution for the Liénard equation is investigated, and a result which improves the classical Dragilev's theorem is presented, together with a corollary in which there are no assumptions on the function $g(x)$, and hence on $G(x)$, besides the standard sign condition. In the final part of the paper constructive examples with several limit cycles are provided.



## 1. Preliminaries and well known results

In this paper we discuss the problem of existence of periodic solutions for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

Such a problem has been widely investigated since the first results of Liénard [17], appeared in 1928 and there is an enormous quantity of papers on this topic.

It is well known that Liénard equation is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.na.2015.06.026
0362-546X/

in the phase plane, and to the system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.2}$$

in the Liénard plane, where $F(x) = \int_0^x f(x)dx$.

For this reason, the problem of the existence of periodic solutions is brought back to a problem of existence of limit cycles for the previous systems, and in this paper we work in the environment of Liénard system (1.2).

Among the existence results until 1960, the classical theorems of Filippov [8], Levinson–Smith [16] and Dragilev [5] may be considered as milestones, while in the last decades the number of results is dramatically increasing.

All the results are based on the classical Poincaré–Bendixson theorem, and in order to fulfil the assumptions of this theorem, it is necessary to produce a winding trajectory large enough.

It is interesting to observe that the methods used to attack this problem are basically two.

We can call the first one the "method of energy", because one may consider the Liénard equation as perturbation of the Duffing equation

$$\ddot{x} + g(x) = 0$$

which plays the rôle of the energy.

Let us discuss in details this situation.

### 1.1. The method of energy

The Duffing equation is equivalent in both planes to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{1.3}$$

and it is well known that the level curves of the function

$$H(x, y) = \frac{1}{2}y^2 + G(x)$$

where $G(x) = \int_0^x g(x)dx$, are its solutions.

Here we follow the elegant and concise description of Lefschetz [14, p. 266]. If we consider the level curve

$$\frac{1}{2}y^2 + G(x) = K \tag{1.4}$$

in the dynamical interpretation as motion of a particle, the first term represents its kinetic energy and (1.4) expresses the law of conservation of energy as applied to the particle.

For this reason, we may consider the level curves of the function $H(x, y)$ as energy levels.

Coming back to the Liénard system (1.2), consider a generic point

$S = (x_S, y_S)$. Keeping the dynamical interpretation, we can say that this point lies on the level of energy

$$\frac{1}{2}y_S^2 + G(x_S) = K_S.$$

For sake of simplicity, we consider a generic point of the $y$-axis $S = (0, y_S)$, which lies on the level energy $\frac{1}{2}y_S^2 = K_S$.