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Large data existence theory for unsteady flows of fluids with pressure- and shear-dependent viscosities

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A B S T R A C T

A generalization of Navier–Stokes' model is considered, where the Cauchy stress tensor depends on the pressure as well as on the shear rate in a power-law-like fashion, for values of the power-law index $r \in \left(\frac{2d}{d+2}, 2\right]$. We develop existence of generalized (weak) solutions for the resultant system of partial differential equations, including also the so far uncovered cases $r \in \left(\frac{2d}{d+2}, \frac{2d+2}{d+2}\right]$ and $r = 2$. By considering a maximal sensible range of the power-law index r, the obtained theory is in effect identical to the situation of dependence on the shear rate only.

1. Introduction

Let $T > 0$, $\Omega \in \mathbb{R}^d$ be an open Lipschitz domain and denote $Q = (0, T) \times \Omega$. We would like to study unsteady flows of incompressible homogeneous fluids in Ω. Setting density to be identically one for simplicity, balance of linear momentum and balance of mass for such fluids can be written down as

$$\begin{aligned} &\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{T} = \boldsymbol{f}, \\ &\operatorname{div} \boldsymbol{v} = 0, \end{aligned} \tag{1}$$

both holding in Q, where $\boldsymbol{f}$ represents the external forces acting on the fluid and $\boldsymbol{T}$ is the Cauchy stress tensor. When the fluid is additionally supposed to be Newtonian, the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \nu \boldsymbol{D}\boldsymbol{v}, \tag{2}$$

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where p is the pressure (the indeterminate part of the stress),

$$\boldsymbol{D}\boldsymbol{v} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right)$$

is the symmetric part of the velocity gradient and $\nu > 0$ is the shear viscosity. When $\boldsymbol{T}$ is of the form (2), Eq. (1) becomes the notorious Navier–Stokes model. Unfortunately, despite all the rapt attention that this model has drawn in renown mathematicians throughout the last century and beyond, the hitherto obtained results are still far from satisfactory. Worse yet, it is well known that this model is incapable of capturing manifold features manifested by non-Newtonian fluids, such as shear-thinning or -thickening, pressure-dependent viscosity etc.

In this paper we are interested in the situation where the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) = -p\boldsymbol{I} + \nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{3}$$

in which the viscous stress tensor $\boldsymbol{S}$ is supposed to meet certain requirements; see Assumptions 2.1 and 2.2. This particular model goes back to two papers by Málek et al. [23,24] and has been dealt with on multiple occasions ever since (see e.g. [5,14,17,22] and the discussion below Theorem 3.1).

It has been convincingly documented in experiments that viscosity of a fluid may vary significantly with the pressure (exponentially or even more dramatically; see e.g. [1,3] or comprehensive references in [27]). Likewise, the already mentioned shear-thinning or shear-thickening behavior can be captured through a non-constant viscosity $\nu = \nu(|\boldsymbol{D}\boldsymbol{v}|^2)$ like in the mathematically popular model of Ladyzhenskaya's. By means of the constitutive relation (3), we can capture both these dependences in a single model. It comes at a price, sadly, for instance we are able to handle only shear-thinning, not shear-thickening, behavior (see the main result, Theorem 3.1, and the upper bound for the power exponent r).

The objective we set is to prove existence of weak solutions for the model. Therefore we have to add initial and suitable boundary conditions, for which sake let us denote $\Gamma = (0, T) \times \partial\Omega$. We consider an impermeable boundary, that is

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

where $\boldsymbol{n}$ is the unit outer normal vector of Ω. We cannot, however, resort to the no-slip boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

for in that case we would be unable to construct the pressure (see the discussion below Theorem 3.1). Instead, we choose the Navier slip condition

$$\alpha \boldsymbol{v}_\tau = -(\boldsymbol{S}\boldsymbol{n})_\tau \quad \text{on } \Gamma$$

for some $\alpha \geq 0$, which is the heart of the matter here due to the dependence of $\boldsymbol{S}$ on p. For $\boldsymbol{u} : \partial\Omega \to \mathbb{R}^d$, a vector field on the boundary, we define its tangential component as

$$\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}.$$

Note that from an instinctive point of view, the Navier slip may be regarded as a bridge between the no-slip condition ($\alpha \to \infty$) and the perfect slip condition ($\alpha = 0$).

On account of the pressure-dependent viscous stress, we have yet to add some kind of pressure anchoring, which we take in the form

$$\frac{1}{|\Omega|} \int_\Omega p(t, x)\, dx = h(t) \quad \text{in } (0, T) \tag{4}$$

for a given function h. Ideally one should like to prescribe the pressure locally (at some point) but since our pressure will be merely an integrable function, dictating its pointwise values is out of the question. A possible