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Large time behavior of solutions to the compressible Navier–Stokes equations around a parallel flow in a cylindrical domain

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A B S T R A C T

Stability of parallel flow of the compressible Navier–Stokes equation in a cylindrical domain is studied. It is shown that if the Reynolds and Mach numbers are sufficiently small, then parallel flow is asymptotically stable and the asymptotic leading part of the disturbances is described by a one dimensional viscous Burgers equation.

1. Introduction

This paper studies the large time behavior of solutions of the initial boundary value problem for the compressible Navier–Stokes equation

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div} v + \nabla p(\rho) = \rho g, \tag{1.2}$$

$$v|_{\partial D_*} = 0, \tag{1.3}$$

$$(\rho, v)|_{t=0} = (\rho_0, v_0) \tag{1.4}$$

in a cylindrical domain $\Omega_* = D_* \times \mathbf{R}$:

$$\Omega_* = \{x = (x', x_3);\, x' = (x_1, x_2) \in D_*, x_3 \in \mathbf{R}\}.$$

Here D_* is a bounded and connected domain in $\mathbf{R}^2$ with smooth boundary ∂D_*; $\rho = \rho(x,t)$ and $v = {}^T(v^1(x,t), v^2(x,t), v^3(x,t))$ denote the unknown density and velocity, respectively, at time $t \geq 0$ and

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position $x \in \Omega_*$; $p(\rho)$ is the pressure that is a smooth function of ρ and satisfies

$$p'(\rho_*) > 0$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that satisfy

$$\mu > 0, \quad \tfrac{2}{3}\mu + \mu' \geq 0;$$

and g is an external force of the form $g = {}^T\big(g^1(x'), g^2(x'), g^3(x')\big)$ with g^1 and g^2 satisfying

$$\big(g^1(x'), g^2(x')\big) = \big(\partial_{x_1}\Phi(x'), \partial_{x_2}\Phi(x')\big),$$

where Φ and g^3 are given smooth functions of x'. Here and in what follows ${}^T\cdot$ stands for the transposition.

Problem (1.1)–(1.3) has a stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s(x'), \overline{v}_s(x'))$ which represents parallel flow. Here $\overline{\rho}_s$ is determined by

$$\begin{cases} \text{Const.} - \Phi(x') = \displaystyle\int_{\rho_*}^{\overline{\rho}_s} \frac{p'(\eta)}{\eta} d\eta, \\ \displaystyle\int_{D_*} \overline{\rho}_s - \rho_* dx' = 0, \end{cases}$$

while $\overline{v}_s$ takes the form

$$\overline{v}_s = {}^T\big(0, 0, \overline{v}_s^3(x')\big),$$

where $\overline{v}_s^3(x')$ is the solution of

$$\begin{cases} -\mu\Delta'\overline{v}_s^3 = \overline{\rho}_s g^3, \\ \overline{v}_s^3\,|_{\partial D_*} = 0. \end{cases}$$

Here

$$\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

The purpose of this paper is to investigate the large time behavior of solutions to problem (1.1)–(1.4) when the initial value $(\rho, v)\,|_{t=0} = (\rho_0, v_0)$ is sufficiently close to the stationary solution $\overline{u}_s = {}^T(\overline{\rho}_s, \overline{v}_s)$.

Solutions of multi-dimensional compressible Navier–Stokes equations in unbounded domains exhibit interesting phenomenon, and detailed descriptions of large time behavior of solutions have been obtained. See, e.g., [6,7,14,17–21,23,24] for the case of the whole space, half space and exterior domains. Besides these domains, infinite layers and cylindrical domains provide good subjects of the stability of flows, for example, the stability of parallel flows.

Concerning the large time behavior of solutions around parallel flow, the case of an n dimensional infinite layer $\mathbf{R}^{n-1} \times (0,1) = \{x = (x_h, x_n);\ x_h = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < 1\}$ was studied in [10,11,15]. (See also [3–5] for the stability of time periodic parallel flow.) It was shown in [10,11,15] that if the Reynolds and Mach numbers are sufficiently small, then the parallel flow is stable under sufficiently small initial disturbances in some Sobolev space. Furthermore, in the case of $n \geq 3$, the disturbance $u(t)$ behaves like a solution of an $n-1$ dimensional linear heat equation as $t \to \infty$, whereas, in the case of $n = 2$, $u(t)$ behaves like a solution of a one dimensional viscous Burgers equation.

In the case of cylindrical domain Ω_*, Iooss and Padula [8] considered the linearized stability of parallel flow $\bar{u}_s$ under periodic boundary condition in x_3. It was proved in [8] that if the Reynolds number is suitably small, then the semigroup decays exponentially as $t \to \infty$, provided that the density-component has vanishing average over the basic period domain. Furthermore, the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\mathrm{Re}\lambda = -c$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities.