Nonlinear Analysis 123–124 (2015) 1–22

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

The free boundary of variational inequalities with gradient constraints

Mohammad Safdari

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA

ARTICLE INFO

Article history:
Received 5 January 2015
Accepted 10 April 2015
Communicated by Enzo Mitidieri

Keywords:
Free boundary
Variational inequality
Gradient constraint
Ridge

ABSTRACT

In this paper we prove that the free boundary of the minimizer of

$$I(v) := \int_U \frac{1}{2}|Dv|^2 - \eta v \, dx,$$

subject to the pointwise gradient constraint

$$|Dv|_p \leq 1,$$

is as regular as the tangent bundle of the boundary of the domain. To this end, we study a generalized notion of ridge of a domain in the plane, which is the set of singularity of the distance function in the p-norm to the boundary of the domain.

1. Introduction

Let U be a bounded, open set in $\mathbb{R}^2$ whose boundary is at least C^2. Suppose $K \subset \mathbb{R}^2$ is a balanced (symmetric with respect to the origin) closed convex set whose interior contains the origin. Let

$$\gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}$$

be the gauge function of K. Also let

$$K^\circ := \{x \mid x \cdot k \leq 1 \text{ for all } k \in K\}$$

be the polar of K. By our assumptions, γ_K and γ_{K° are norms on $\mathbb{R}^2$ (see Rockafellar [1]). Let d_{K° be the metric associated to the norm γ_{K°.

We assume that γ_{K° is strictly convex. This is equivalent to requiring K° to be strictly convex, i.e. its boundary does not contain any line segment. The latter happens, for example, when K is strictly convex

E-mail address: safdari@gmail.com.

http://dx.doi.org/10.1016/j.na.2015.04.009

and its boundary is C^1 (see Crasta and Malusa [2]). As an example, let $K = \{x \mid \gamma_q(x) \le 1\}$ for $q > 1$, where $\gamma_q(x) = |x|_q := (|x_1|^q + |x_2|^q)^{1/q}$ is the q-norm of $x = (x_1, x_2)$. Then $\gamma_{K^\circ} = \gamma_p$ is strictly convex, where $p = \frac{q}{q-1}$ is the dual exponent to q.

Let

$$I(v) := \int_U \frac{1}{2}|Dv|^2 - \eta v\, dx,$$

where $\eta > 0$. Let u be the minimizer of I over

$$W_K := \{v \in H_0^1(U) \mid \gamma_K(Dv) \le 1 \text{ a.e. }\}.$$

It can be shown that u is also the minimizer of I over

$$\{v \in H_0^1(U) \mid v(x) \le d_{K^\circ}(x, \partial U) \text{ a.e. }\}.$$

For the proof see Brezis and Sibony [3], Treu and Vornicescu [4] and Safdari [5].

When K is the unit disk, and therefore $\gamma_K, \gamma_{K^\circ}$ are both the Euclidean norm, the above problem is the famous elastic–plastic torsion problem. The regularity of the free boundary of elastic–plastic torsion problem is studied by Caffarelli and Rivière [6], Caffarelli and Rivière [7], Caffarelli and Friedman [8], Friedman and Pozzi [9], Caffarelli et al. [10]. Their work is explained by Friedman [11].

In this paper, we extend their results to the more general problem explained above. A motivation for our study was to fill the gap between the known regularity results mentioned above and the still open question of regularity of the minimizer of some convex functionals subject to gradient constraints arising in random surfaces. To learn about the latter see the work of De Silva and Savin [12].

In order to study the free boundary, we generalize the notion of ridge, by replacing the Euclidean norm by other norms. We also consider the singularities of the distance function (in the new norm) to the boundary of a domain. These notions have been considered and applied before by Li and Nirenberg [13], Crasta and Malusa [2]. In particular the work of Crasta and Malusa [2] has considerable intersection with ours. The difference between our work and theirs lies in that we allow less regular domains, and consider norms that are less restricted in some aspects than what they consider.

2. The ridge

First, we start by generalizing the notion of ridge.

Definition 1. The K-**ridge** of U is the set of all points $x \in U$ where

$$d_K(x) := d_K(x, \partial U)$$

is not $C^{1,1}$ in any neighborhood V of x. We denote it by

$$R_K.$$

Lemma 1. *Suppose γ_K is strictly convex. If $d_K(x) = \gamma_K(x - y) = \gamma_K(x - z)$ for two different points y, z on ∂U, then d_K is not differentiable at x.*

Proof. Along the segment $\overline{xy}$ (and similarly $\overline{xz}$) we have

$$\begin{aligned} d_K\big(x + \frac{t}{\gamma_K(x-y)}(y - x)\big) &= \gamma_K\big(x + \frac{t}{\gamma_K(x-y)}(y - x) - y\big) \\ &= \gamma_K(x - y) - t. \end{aligned} \tag{2.1}$$