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Stability of eigenvalues for variable exponent problems

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ABSTRACT

In the framework of variable exponent Sobolev spaces, we prove that the variational eigenvalues defined by inf sup procedures of Rayleigh ratios for the Luxemburg norms are all stable under uniform convergence of the exponents.

1. Introduction and main result

The differential equations and variational problems involving $p(x)$-growth conditions arise from nonlinear elasticity theory and electrorheological fluids, and have been the target of various investigations, especially in regularity theory and in nonlocal problems (see e.g. [1–3,10,16,32] and the references therein). Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a bounded domain and let $p : \bar{\Omega} \to \mathbb{R}^+$ be a continuous function such that

$$1 < p_- := \inf_{\Omega} p \leq p(x) \leq \sup_{\Omega} p =: p_+ < N \quad \text{for all } x \in \Omega. \tag{1.1}$$

We also assume that p is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq -\frac{L}{\log |x - y|} \tag{1.2}$$

for some $L > 0$ and for all $x, y \in \Omega$, with $0 < |x - y| \leq 1/2$. From now on, we denote by

$$\mathscr{C} := \left\{ p \in C(\bar{\Omega}) : p \text{ satisfies (1.1) and (1.2)} \right\}$$

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the set of admissible variable exponents. The goal of this paper is to study the stability of the (variational) eigenvalues with respect to (uniform) variations of $p(\cdot)$ for the problem

$$-\operatorname{div}\Big(p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)-2}\frac{\nabla u}{K(u)}\Big)=\lambda S(u)p(x)\left|\frac{u}{k(u)}\right|^{p(x)-2}\frac{u}{k(u)},\quad u\in W_0^{1,p(x)}(\Omega), \tag{1.3}$$

where we have set

$$K(u):=\|\nabla u\|_{p(x)},\qquad k(u):=\|u\|_{p(x)},\qquad S(u):=\frac{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)}dx}{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)}dx}.$$

Following the argument contained in [21, Section 3], it is possible to derive Eq. (1.3) as the Euler–Lagrange equation corresponding to the minimization of the Rayleigh ratio

$$\frac{K(u)}{k(u)}=\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}},\quad \text{among all } u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}, \tag{1.4}$$

where $\|\cdot\|_{p(x)}$ denotes the Luxemburg norm of the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ (see Section 2). This minimization problem has been firstly introduced in [21] as an appropriate replacement for the *inhomogeneous* minimization problem

$$\frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}dx}{\displaystyle\int_\Omega |u|^{p(x)}dx},\quad \text{among all } u\in W_0^{1,p(x)}(\Omega)\setminus\{0\},$$

which was previously considered in [20] to define the first eigenvalue λ_1 of the $p(x)$-Laplacian. In [20], sufficient conditions for λ_1 defined in this way to be zero or positive are provided. In particular, if $p(\cdot)$ has a strict local minimum (or maximum) in Ω, then $\lambda_1=0$. Arguing as in [21, Lemma A.1], it can be shown that the functionals k and K are differentiable with

$$\langle K'(u),v\rangle=\frac{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)-2}\frac{\nabla u}{K(u)}\cdot\nabla v\,dx}{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)}dx}\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega),$$

$$\langle k'(u),v\rangle=\frac{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)-2}\frac{u}{k(u)}v\,dx}{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)}dx}\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega).$$

Therefore, all the critical values of the quotient (1.4) are eigenvalues of Eq. (1.3) and vice versa. The m-th (variational) eigenvalue $\lambda^{(m)}_{p(x)}$ of (1.3) can be obtained as

$$\lambda^{(m)}_{p(x)}:=\inf_{K\in\mathcal{W}^{(m)}_{p(x)}}\sup_{u\in K}\|\nabla u\|_{p(x)},$$

where $\mathcal{W}^{(m)}_{p(x)}$ is the set of symmetric, compact subsets of $\{u\in W_0^{1,p(x)}(\Omega):\|u\|_{p(x)}=1\}$ such that $i(K)\ge m$, and i denotes the Krasnosel'skiĭ genus (or, actually, any other index satisfying the properties listed in Remark 1.4). In [21] existence and properties of the first eigenfunction were studied, while in [7] a numerical method to compute the first eigenpair of (1.3) was obtained and the symmetry breaking phenomena with