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Existence and structure of steady solutions of the Bénard problem in a two dimensional quadrangular cavity

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ABSTRACT

We prove the existence of a strong–weak solution $(\mathbf{u}, p, T)$ (= velocity, pressure, temperature) of the steady Bénard problem in a 2D quadrangular cavity, heated/cooled on two opposite sides and thermally insulated on the other sides. Applying the tools of nonlinear analysis, we study the structure of the set of solutions in dependence on the acting volume force and on the given temperature profiles on the heated/cooled sides. Particularly, in the case when the cavity has the form of a trapezoid, we also study the structure of the solution set in dependence on the angle of inclination from the horizontal–vertical position.

1. Introduction, formulation of the boundary-value problem

1.1. Motivation

Convection in a differentially heated rectangular box with insulated side walls sloping at an angle φ to the horizontal with its limiting cases of Rayleigh–Bénard convection ($\varphi = 0$) and convection in the vertical position ($\varphi = \pi/2$) has been of interest to researchers given its many applications both from the mathematical physics and engineering science points of view starting in the seventies with Hart [19] and continuing to this day with the most recent papers appearing only a few years ago, Lyes et al. [24,25]. Up to the pioneering stability analysis of Hart [19] work was done only on zero aspect ratio geometries. The case of convection between horizontal parallel plates driven by temperature gradients due to the hot and cooled

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bottom and top plates, respectively, is well known and has been critically studied from an engineering science perspective, Davis [8], Segel [35], among others, as well as from a rigorous mathematical point of view by Rabinowitz [34]. Convection between vertical parallel plates had been studied earlier by Batchelor [4]. However flow in a non-zero aspect ratio enclosure is considerably more complicated including new types of instabilities as the structure of the mean flow is significantly changed due to the varying buoyancy force components with the inclination angle. For instance it is well established now that flow instabilities exist in the inclined box system as both stationary, Vest and Arpaci [38], and traveling disturbances, Gill and Davey [16]. It is important to determine to what extent the instabilities in the inclined convective case resemble those occurring in the horizontal and vertical limits. Inclined geometry introduces phenomena of rather profound significance, which do not occur in either of the limits.

De Graaf and Van der Held [9] were the first to observe longitudinal instabilities in a convecting layer slightly inclined from the vertical. However they did not offer any physical explanation. The experimental study of Dropkin and Somerscales [10] is to our knowledge the first study of heat transfer in an inclined enclosure. Starting with Hollands and Konicek [20], Catton et al. [6] and Arnold et al. [1] flow pattern change with varying inclination in a differentially heated box was identified. This complex phenomenon of stability exchange of flow pattern at critical angles of inclination which depends on the aspect ratio of the box and the physical (constitutive) parameters of the fluid as well as the temperature difference of the isothermally maintained heated and cooled walls in two dimensional enclosures was studied intensively in a series of papers by Ozoe et al. [31–33] (of which we listed only a few here) without a conclusive agreement between numerical/theoretical predictions and experimental results. Various researchers since then focused their efforts on clarifying the physical underpinnings of the complexities of the flow and heat transfer in this simple geometry, Soong et al. [37] and Lyes et al. [24,25] among others.

Selected reviews and references can offer an overall view of the gradual progress made over several decades and the present state of the science in this area and the still unresolved features of the fundamental problem. The earliest reviews by Ostrach [29] followed by Catton [5] in the seventies describe the state of the science on this problem up to that time. More recently the reviews by Baïri [2] and Fusegi and Hyun [13], among others, point out the complexities associated with the problem. Fusegi and Hyun [13] for example consider the effect of the spatial and temporal variations of thermal boundary conditions as well as variable physical property effects and multi-dimensionalities. The problem of the flow and heat transfer in the three dimensional enclosure brings about an additional layer of complexity as described in the relatively recent work of Baïri et al. [3] although there were earlier attempts to treat it, Ozoe et al. [30].

In comparison with the huge number of physically oriented works, there do not exist many rigorous mathematical treatments of the general problem of the Rayleigh–Bénard convection. Here we cite the papers of Rabinowitz [34] and Morimoto [26,27]. However their work is related to special cases of the general problem as explained more in detail in Section 1.7. In this paper, we present a mathematical theory related to the general problem, and in addition to the basic existential theory for the weak–strong solutions (weak in the velocity part and strong in the temperature part), we also pay attention to the qualitative description of the solution set, its structure and the way it depends on various data of the problem.

1.2. The flow domain and the structure of the volume force

We study the 2D Bénard problem in a two-dimensional convex quadrangular cavity $A_1A_2B_2B_1$, see Fig. 1. The interior angles at the vertices A_1 and A_2 are denoted by ω_1 and ω_2. The Cartesian system of coordinates is chosen so that its origin is at point A_1, the side A_1A_2 lies on the x_1-axis and the quadrangle $A_1A_2B_2B_1$ lies in the half-plane $x_2 \geq 0$. We denote by Ω the interior of the quadrangle $A_1A_2B_2B_1$ and by d the maximum of the x_2 coordinates of points B_1, B_2.