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On the structure of conformally flat Riemannian manifolds ☆

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In this paper we will prove vanishing and finiteness theorems for L^2-harmonic 1-forms on a locally conformally flat Riemannian manifold which satisfies an integral pinching condition on the traceless Ricci tensor, and for which the scalar curvature is non-positive or satisfies some integral pinching conditions. Based on these vanishing and finiteness theorems, combining with the work of Li–Tam, we can obtain some one-end and finite ends theorems.

1. Introduction

Let us recall that an n-dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat (LCF for short) if it admits a coordinate covering $\{U_\alpha, \varphi_\alpha\}$ such that the map $\varphi_\alpha : (U_\alpha, g_\alpha) \to (S^n, g_0)$ is a conformal map, where g_0 is the standard metric on S^n. It is well known that a conformally flat manifold is a higher dimensional generalization of a Riemannian surface. But not every higher dimensional manifold admits a locally conformally flat structure, and it is difficult to give a good classification of LCF Riemannian manifolds. However, by adding various geometric conditions, many authors have given some partial classification for LCF Riemannian manifolds (see, for example, [4,6,5,8,7,9,14,16,19], etc.).

It is interesting to investigate the topological structure of LCF manifolds under suitable geometric restrictions. In [15], Pigola, Rigoli and Setti proved a vanishing theorem for bounded harmonic m-forms on a $2m$-dimensional complete LCF manifold by putting some assumptions on scalar curvature and volume growth. In [13], Li–Wang showed that for a complete, simply connected, LCF manifold M^n $(n \geq 4)$ with scalar curvature $R \geq 0$, if the Ricci curvature $\mathrm{Ric} \geq \frac{1}{4}R$ and the scalar curvature satisfies some decay condition, then either M has only one end, or $M = \mathbb{R} \times N$ with a warped product metric for some compact manifold N.

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For a compact Riemannian manifold M^n, according to Hodge theory, the space of harmonic 1-forms on M^n is isomorphic to its first de Rham cohomology group. And it is well known that there are no harmonic p-forms, $0 < p < n$, on a compact conformally flat manifold M^n of positive Ricci curvature. When M is non-compact, the Hodge theory does not work anymore, hence it is natural to consider L^2-harmonic forms, as is showed that L^2-Hodge theory remains valid in complete non-compact manifolds as classical Hodge theory works well in the compact case. In particular, it is shown by Li–Tam [11] that the theory of L^2-harmonic 1-forms can be used to study the topology at infinity of a complete Riemannian manifold. Let $H^1(L^2(M))$ be the space of L^2-harmonic 1-forms on M. It is well known that the space $H^1(L^2(M))$ is isomorphic to the first-reduced L^2-cohomology group of M. For more results concerning L^2-harmonic forms on complete non-compact manifolds, one can consult a very nice survey of G. Carron [3].

We denote by Ric and R the Ricci and the scalar curvature, respectively and by T the traceless Ricci tensor, i.e., $T = \mathrm{Ric} - \frac{R}{n}g$, then $|T|^2 = |\mathrm{Ric}|^2 - \frac{R^2}{n}$. According to the decomposition of the Riemannian curvature, it is obvious that a locally conformally flat manifold M has constant sectional curvature if and only if it is Einstein, that is, $T = 0$. Moreover, it is well known that $H^1(L^2(M)) = \{0\}$, if M is a space form, which means that $T = 0$. Hence the following question arises naturally: If M is close enough to the space form in the integral sense, is the conclusion $H^1(L^2(M)) = \{0\}$ still valid? In this paper, we propose to prove vanishing and finiteness theorems for L^2-harmonic 1-forms on LCF manifolds under appropriate integral assumptions on T.

When the scalar curvature of a LCF Riemannian manifold M^n has finite $L^{\frac{n}{2}}$-norm, then the L^2 Sobolev inequality holds on M^n. Using the Sobolev inequality, we can apply the standard Moser iteration to the elliptic inequality with respect to harmonic 1-forms. Combining with Peter Li's lemma [10], one can prove finiteness theorems for L^2-harmonic 1-forms, under the assumption that the $L^{n/2}$-norm of the traceless Ricci tensor T is finite. Finally, according to the results of [11,12], we have the following finite ends theorem.

Theorem 1.1. *Let (M^n, g), $n \geq 3$, be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that*

$$\int_M |R|^{\frac{n}{2}} dv < \infty, \tag{1.1}$$

and

$$\int_M |T|^{\frac{n}{2}} dv < \infty. \tag{1.2}$$

Then $\dim H^1(L^2(M)) < \infty$. In particular, M must have finitely many ends.

When the scalar curvature of a LCF manifold M^n is non-positive, then the L^2 Sobolev inequality and the weighted Poincaré inequality hold on M^n. In this case, we will deduce the following finite ends theorem only under the assumption that $\|T\|_{L^{\frac{n}{2}}(M)}$ is finite.

Theorem 1.2. *Let (M^n, g), $n \geq 17$, be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$. Assume that*

$$\int_M |T|^{\frac{n}{2}} dv < \infty.$$

Then $\dim H^1(L^2(M)) < \infty$. In particular, M must have finitely many ends.

Finally, if the geometrical invariants are less than explicit constants, we will prove vanishing theorems for L^2-harmonic 1-forms. Using these vanishing theorems and the work of [11,12], we will obtain the following one-end theorems.