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Sections of stable harmonic convex functions

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ABSTRACT

In this article, we consider the class $\mathcal{SHC}$ of normalized stable harmonic convex mappings $f = h + \overline{g}$ in the unit disk and determine r such that every section of f is close-to-convex or convex in the disk $|\, z \,| < r$. Also, we show that the convolution $f * f \in \mathcal{SHC}$ whenever $f \in \mathcal{SHC}$. In addition, we prove that the harmonic convolution $f_1 * f_2$ is stable harmonic close-to-convex whenever $f_1 \in \mathcal{SHC}$ and f_2 is either a slanted half-plane mapping or an asymmetric vertical strip mapping or a mapping in the family $\mathcal{P}_H^0(\alpha)$ which is contained in the family of harmonic close-to-convex functions in $|\, z \,| < 1$.

1. Introduction

For $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$, the open unit disk. Let $\mathcal{H}$ consist of all complex-valued harmonic functions $f = h + \overline{g}$ defined on $\mathbb{D}$, where h and g are analytic on $\mathbb{D}$ such that $h(0) = 0 = h'(0) - 1$ and $g(0) = 0$. Clearly, each $f = h + \overline{g} \in \mathcal{H}$ has the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

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A necessary and sufficient condition for $f = h + \overline{g} \in \mathcal{H}$ to be sense-preserving in $\mathbb{D}$ is that the Jacobian $J_f(z)$ is positive in $\mathbb{D}$, where $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. That is, there exists an analytic function $\omega(z) = g'(z)/h'(z)$, called the dilatation of f, such that $|\omega(z)| < 1$ for $z \in \mathbb{D}$.

Also, let $\mathcal{H}_0 = \{f = h + \overline{g} \in \mathcal{H} : g'(0) = 0\}$, and let $\mathcal{S}_H$ denote the subclass of $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$, and further set $\mathcal{S}_H^0 = \mathcal{S}_H \cap \mathcal{H}_0$. Recall that both $\mathcal{S}_H$ and $\mathcal{S}_H^0$ coincide with the class $\mathcal{S}$ of the classical normalized analytic univalent mappings, whenever the co-analytic part g of $f = h + \overline{g}$ is identically zero. Important geometric subclasses of $\mathcal{S}_H$ and $\mathcal{S}_H^0$ such as convex, close-to-convex, starlike (with respect to the origin) and typically real harmonic functions are discussed by Clunie and Sheil-Small [1] and these classes were investigated later by a number of authors. For many interesting results and expositions on planar harmonic univalent mappings, we refer the book of Duren [2] and also the articles [3,4]. However, the class $\mathcal{S}_H^0$ is the central object in the study of harmonic univalent mappings.

Lemma A ([1]). *If a harmonic mapping $f = h + \overline{g}$ on $\mathbb{D}$ satisfies $|g'(0)| < |h'(0)|$ and the function $F_\lambda = h + \lambda g$ is close-to-convex for all $|\lambda| = 1$, then f is close-to-convex and univalent in $\mathbb{D}$.*

Lemma A due to Clunie and Sheil-Small [1] motivates one to introduce and study stable harmonic mappings which we now recall (see [5]): A sense-preserving harmonic mapping $f = h + \overline{g} \in \mathcal{H}_0$ is *stable harmonic convex* (resp. *stable harmonic close-to-convex*) in $\mathbb{D}$ if all the mappings $f_\lambda = h + \lambda g$ are convex (resp. close-to-convex) in $\mathbb{D}$, where $|\lambda| = 1$. The set of all stable harmonic convex (resp. stable harmonic close-to-convex) mappings is denoted by $\mathcal{SHC}$ (resp. $\mathcal{SHCC}$). For example, the following lemma is easy to obtain.

Lemma B ([5, Proposition 8.2]). *Suppose that $f = h + \overline{g} \in \mathcal{SHC}$, where h and g are in the form (1). Then for any $n \geq 2$,*

1. $|a_n + \lambda b_n| \leq |a_n| + |b_n| \leq 1$, *where* $|\lambda| = 1$;
2. $|a_n| \leq 1$.

All the results are sharp, with $f(z) = \frac{z}{1-z}$ being extremal.

For $f = h + \overline{g} \in \mathcal{H}_0$, where h and g are in the form (1), we define the sections/*partial sums* of h, g and f as follows:

$$s_p(h)(z) = \sum_{k=1}^{p} a_k z^k, \qquad s_q(g)(z) = \sum_{k=2}^{q} b_k z^k, \qquad s_{p,q}(f) = s_p(h) + \overline{s_q(g)},$$

where $a_1 = 1, p \geq 1$ and $q \geq 2$. One of the classical results of Szegö [6] (see also [7, Theorem 8.5]) shows that if $h \in \mathcal{S}$ is defined by (1), then the nth *partial sums* $s_n(h)$ is univalent in $|z| < 1/4$ and the number $1/4$ cannot be replaced by a larger one. Although every section of the Koebe function $k(z)$ is univalent in the disk $|z| < 1 - 3n^{-1} \log n$ for $n \geq 5$ and that the constant 3 cannot be replaced by a smaller number, Bshouty and Hengartner [8, p. 408] have observed that the Koebe function is not extremal for the problem of determining the radius of univalency of the partial sums of $h \in \mathcal{S}$. Thus, the largest radius of univalence r_n of $s_n(h)$ $(h \in \mathcal{S})$ is not yet known although the conjectured value is $1 - 3n^{-1} \log n$ for $n \geq 5$. The reader is referred to [9–13] for many interesting results on sections of various subclasses of $\mathcal{S}$.

Lemma C ([11,6]). *Let $h \in \mathcal{S}$ be convex (resp. starlike, close-to-convex) in $\mathbb{D}$. Then $s_n(h)$ is convex (resp. starlike, close-to-convex) in $|z| < 1/4$ for all $n \geq 2$. Moreover, $s_n(h)$ is convex (resp. starlike, close-to-convex) in $|z| < 1 - 3n^{-1} \log n$ for $n \geq 5$.*

In fact, Lemma C is a consequence of a convolution theorem due to Ruscheweyh and Sheil-Small [14] and the fact that the section $s_n(h)$ of $h(z) = z/(1 - z)$ is convex in the disk $|z| < 1 - 3n^{-1} \log n$ for $n \geq 5$.