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Existence of positive ground state solutions for Kirchhoff type problems ☆

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ABSTRACT

In this paper, we study the existence of positive ground state solutions for the nonlinear Kirchhoff type problem

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\triangle u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\, u>0 & \text{in } \mathbb{R}^3,\end{cases}$$

where $a, b > 0$ are constants, $f \in C(\mathbb{R}, \mathbb{R})$ is subcritical near infinity and superlinear near zero and satisfies the Berestycki–Lions condition. By using an abstract critical point theorem established by Jeanjean and a new global compactness lemma, we show that the above problem has at least a positive ground state solution. Our result generalizes the results of Li and Ye (2014) concerning the nonlinearity $f(u) = |u|^{p-1}u$ with $p \in (2, 5)$.

1. Introduction

In this paper, we deal with the following Kirchhoff-type problem:

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\triangle u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\qquad u>0 & \text{in } \mathbb{R}^3,\end{cases} \tag{K}$$

where $a, b > 0$ are constants and $f \in C(\mathbb{R}, \mathbb{R})$.

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$\begin{cases} u_{tt}-\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\right)\triangle u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, u denotes the displacement, $f(x, u)$ is the external force and b is the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed

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by Kirchhoff [19] in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. Since an abstract functional analysis framework was introduced by Lions [22], (1.1) has been receiving much attention (see, for example, [13,12,5,11,26] and the references therein).

We remark that the stationary problem associated with (1.1) is

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\triangle u = f(x,u) & \text{in } \Omega,\\ u\in H^1(\mathbb{R}^3) & \text{in } \Omega,\end{cases} \tag{1.2}$$

and has been studied by many researchers by using variational methods; see, for example, [11,2,3,33,27,10,9,14,21] and the references therein. More precisely, Ma and Rivera [26] proved the existence and nonexistence of positive solutions for a class of Kirchhoff type systems via variational methods. The existence of positive solutions was also proved in [3,9]. Perera and Zhang [29] and Mao [27] used minimax methods and invariant sets of descent flow to investigate the existence of sign changing solutions of (1.2).

We also note that there are many existence results for the following Kirchhoff type problem on $\mathbb{R}^N$

$$\begin{cases} -\left(\epsilon^2 a+b\epsilon\int_{\mathbb{R}^3}|\nabla u|^2\right)\triangle u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^3) & \text{in } \mathbb{R}^N,\end{cases} \tag{1.3}$$

where $N = 1, 2$ or 3. See, for example, [32,18,15,30,28,24,25]. In particular, Wu [31] obtained a sequence of high energy solutions for problem (1.3) with $\epsilon = 1$ via a symmetric Mountain Pass Theorem under the condition that $f(x, u)$ is 4-suplinear at infinity and satisfies that

(AR) There exists $\theta > 4$ such that $0 < \theta F(x,u) \le f(x,u)u$ for all $u \ne 0$, where $F(x,u) = \int_0^u f(x,t)dt$,

and that the potential $V(x) \in C(\mathbb{R}^N,\mathbb{R})$ satisfies

$$\inf_{x\in\mathbb{R}^N} V(x) \ge a_1 > 0, \quad \text{meas}\{x\in\mathbb{R}^N : V(x)\le M\} < +\infty \text{ for each } M > 0.$$

The condition about the potential $V(x) \in C(\mathbb{R}^N,\mathbb{R})$ is used by Wu [31] to ensure the compactness of embedding of

$$E = \left\{u\in H^1(\mathbb{R}^3)\,\middle|\,\int_{\mathbb{R}^3}V(x)|u|^2 < \infty\right\} \hookrightarrow L^s(\mathbb{R}^3), \quad 2\le s<6.$$

When $V(x)$ is radially symmetric, Nie and Wu [28] also studied the existence of infinitely many high energy solutions for problem (1.3) with $\epsilon = 1$ by using the Mountain Pass Theorem. Recently, Alves and Figueiredo [4] studied the following class of Kirchhoff problem on $\mathbb{R}^N$:

$$M\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\right)[-\triangle u+V(x)u] = \lambda f(u)+\gamma u^\tau \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $\tau = 5$ if $N = 3$ and $\tau \in (1, +\infty)$ if $N = 1, 2$; $\lambda > 0$ and $\gamma \in \{0, 1\}$. Under certain conditions on functions M, V and f, Alves and Figueiredo [4] showed that there exists a constant $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, (1.4) with $\gamma = 1$ has at least a positive solution, while for every $\lambda > 0$, (1.4) with $\gamma = 0$ has a positive solution. Li et al. [21] studied the existence of a positive solution for the following Kirchhoff problem in a radial symmetric space

$$\begin{cases} \left(a+\varepsilon\int_{\mathbb{R}^N}|\nabla u|^2+\varepsilon b\int_{\mathbb{R}^N}u^2\right)[-\triangle u+bu] = f(u) & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\ u>0 & \text{in } \mathbb{R}^N,\end{cases} \tag{1.5}$$

where $N \ge 3$, a, b are positive constants, $\varepsilon > 0$ is a parameter and the nonlinearity $f(u)$ satisfies the following conditions:

(H1) $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $|f(u)| \le C(|u| + |u|^{q-1})$ for all $u \in \mathbb{R}^+$ and some $q \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$.

(H2) $\lim_{t\to 0}\frac{f(t)}{t} = 0$.

(H3) $\limsup_{t\to\infty}\frac{f(t)}{t} = +\infty$.

By using a truncation argument combined with a monotonicity trick introduced by Jeanjean [17], Li et al. [21] showed that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0)$, (1.5) has at least one positive radial symmetric solution. However, the method of Li et al. [21] could neither be applied to the case where ε is an arbitrary positive constant nor to obtaining the existence of a ground state solution in $H^1(\mathbb{R}^3)$. When $b = V(x)$ and $f(u) = |u|^{p-1}u$ with $p \in (1, 5)$, Huang and Liu [16] proved that there exist a constant ε^* depending on parameter $p \in (1, 3]$ and a constant $\tilde{\varepsilon}$ independent of p such that (1.5) with $p \in (1, 3]$ and $\varepsilon > \varepsilon^*$ has no nontrivial solution, but (1.5) with $\varepsilon \in (0, \tilde{\varepsilon})$ has two positive solutions for $p \in (1, 3)$ and one positive solution for $p = 3$. Moreover, for (1.5) with $p \in (3, 5)$ and $\varepsilon > 0$, Huang and Liu [16] also obtained the existence of least energy nodal solution with exactly two nodal domains.