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A necessary and sufficient condition for lower semicontinuity

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ABSTRACT

It is well-known that $W^{1,p}$ quasiconvexity is a necessary condition for sequential weak lower semicontinuity of the variational integral

$$I[u,\Omega]=\int_{\Omega}F(\nabla u(x))\,\mathrm{d}x$$

on the Sobolev space $W^{1,p}$, and that it is sufficient too provided that the integrand F satisfies suitable growth conditions related to the exponent p. We show that for extended real-valued integrands a closely related convexity condition defined in terms of gradient Young measures is both necessary and sufficient for lower semicontinuity in the more general – and flexible – setting of compensated compactness. Our main results identify the relaxation (lower semicontinuous envelope) in two related situations.

1. Introduction

For an integrand F defined on $N\times n$ matrices and satisfying the growth conditions

$$\frac{1}{c}|\xi|^p-c\leq F(\xi)\leq c(|\xi|^p+1) \tag{1.1}$$

for all matrices ξ, where $c>0$ is a constant and the exponent $p>1$, it has been shown by Dacorogna (see [13]) that the lower semicontinuous envelope of the integral $I[u,\Omega]$ in the weak topology of $W^{1,p}=W^{1,p}(\Omega,\mathbb{R}^N)$ equals

$$\int_{\Omega}\tilde{F}(\nabla u(x))\,\mathrm{d}x,$$

where $\tilde{F}$ denotes the $W^{1,p}$ quasiconvex envelope of F. We refer to Section 2 for notation and terminology. The results of this paper establish similar relaxation formulas in situations where the integrand F fails to satisfy the stringent growth conditions (1.1), but merely satisfies a (p,q) growth condition for suitable exponents p and q, and, in the compensated compactness setting, is allowed to be extended real-valued. The need to consider more general growth conditions than (1.1) arises for instance in connection with some mathematical models in continuum mechanics (see for instance [3–5,12,15,16,24,28,40]). We also remark that while ordinary convexity and lower semicontinuity of the integrand F is sufficient for (all kinds of) lower semicontinuity properties of the functional $I[u,\Omega]$, in the vectorial multi-dimensional case $n,N>1$ that we consider here, it is very far from necessary [15].

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We proceed to describe and state the main results.

Unless stated otherwise we denote by Ω a nonempty, bounded and open subset of $\mathbb{R}^n$ with Lipschitz boundary. The results established below hold under much weaker assumptions on the boundary, but since we have nothing new to add in that direction we opted for simplicity. We also fix an exponent $p \in (1, \infty)$.

Let $F\colon \mathbb{R}^{N\times n} \to [0, \infty]$ be an extended real-valued Borel integrand. For a matrix field V in $\mathrm{L}^p = \mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$ define

$$\bar{\mathrm{I}}[V] = \inf_{(V_j)} \left\{ \liminf_{j\to\infty} \int_\Omega F(V_j(x))\,\mathrm{d}x \right\} \tag{1.2}$$

where the infimum is taken over all sequences (V_j) of L^p matrix fields on Ω satisfying

$$V_j \rightharpoonup V \quad \text{weakly in } \mathrm{L}^p \quad \text{and} \quad \mathrm{curl}V_j \to \mathrm{curl}V \quad \text{strongly in } \mathrm{W}^{-1,p}. \tag{1.3}$$

Here curl on an $N \times n$ matrix field on Ω is understood row-wise and in the distributional sense, $\mathrm{W}^{-1,p}$ is the dual space of $\mathrm{W}^{1,p'}_0$ under the usual distributional duality pairing and $p' = p/(p-1)$ is the Hölder conjugate exponent. This is a special case of the compensated compactness setting as considered in [14,22,23,42,50], see also [2,8,33]. Our first result concerns functionals in this setting and is the following relaxation formula.

Theorem 1.1. *Let $F\colon \mathbb{R}^{N\times n} \to [0, \infty]$ be a Borel integrand satisfying $F(\xi) \geq \frac{1}{c}|\xi|^p - c$ for all ξ, where $c > 0$ is a constant. Then for each V in $\mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$ the relaxation $\bar{\mathrm{I}}$ defined at (1.2) is given by*

$$\bar{\mathrm{I}}[V] = \int_\Omega \bar{F}(V(x))\,\mathrm{d}x \tag{1.4}$$

where $\bar{F}$ denotes the closed $\mathrm{W}^{1,p}$ quasiconvex envelope of F.

The integrand F is said to be closed $\mathrm{W}^{1,p}$ quasiconvex provided it is lower semicontinuous and Jensen's inequality holds for F and every homogeneous $\mathrm{W}^{1,p}$ gradient Young measure. Then the closed $\mathrm{W}^{1,p}$ quasiconvex envelope of an integrand F is by definition the largest closed $\mathrm{W}^{1,p}$ quasiconvex integrand below F. We refer the reader to Section 2 for a review of notation and terminology.

We record an easy consequence of Theorem 1.1 in the following.

Corollary 1.2. *Let $F\colon \mathbb{R}^{N\times n} \to [0, \infty]$ be Borel, $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be open. Then*

$$\liminf_{j\to\infty} \int_\Omega F(V_j(x))\,\mathrm{d}x \geq \int_\Omega F(V(x))\,\mathrm{d}x \tag{1.5}$$

holds for all sequences (V_j) and V satisfying (1.3) if and only if F is closed $\mathrm{W}^{1,p}$ quasiconvex.

For a Sobolev map u in $W^{1,p}(\Omega, \mathbb{R}^N)$ define

$$\tilde{\mathrm{I}}[\nabla u] = \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega F(\nabla u_j(x))\,\mathrm{d}x \right\} \tag{1.6}$$

where the infimum is taken over all sequences (u_j) in $W^{1,p}(\Omega, \mathbb{R}^N)$ satisfying $u_j \rightharpoonup u$ weakly in $\mathrm{W}^{1,p}$, that is,

$$u_j \rightharpoonup u \quad \text{weakly in } \mathrm{L}^p \quad \text{and} \quad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } \mathrm{L}^p. \tag{1.7}$$

Obviously the sequence (∇u_j) converges to ∇u in the sense of (1.2), so that in particular $\tilde{\mathrm{I}}[\nabla u] \geq \bar{\mathrm{I}}[\nabla u]$ holds for all $\mathrm{W}^{1,p}$ maps u. The following elementary example is based on the T_4 configuration of four matrices (see [44,51]). It shows that the above inequality for relaxed functionals can be strict for lower semicontinuous extended real-valued integrands F.

Example 1.3. Let

$$K = \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \pm\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The following three facts are well-known (see for instance [40]):

(i) There exists a laminate ν which is supported on K and has centre of mass $\bar{\nu} = 0$.
(ii) Laminates with compact support are homogeneous $\mathrm{W}^{1,p}$ gradient Young measures for any p.
(iii) If $\Omega \subset \mathbb{R}^2$ is open and connected, $u \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^2)$ and $\nabla u \in K$ a.e., then ∇u is constant a.e. (See also [11] for far reaching generalizations.)

If therefore we define

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \in K \\ \infty & \text{else,} \end{cases}$$