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# A two obstacles coupled problem

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ABSTRACT

We consider a system of an evolutionary variational inequality of two obstacles type, depending on the temperature, coupled with the heat equation. We prove existence of solution of this system and we present examples that motivated this work. In particular, with additional assumptions on the data, we prove that solutions of this problem are also solutions of a similar problem where the convex set is of gradient constraint type (that depends on the temperature), improving a previous result.



## 1. Introduction

In certain situations, there exists a close relation between a variational inequality with gradient constraint and the same variational inequality with two obstacles. The following well-known variational formulation of the elastic–plastic torsion problem: to find $u \in \mathbb{K}^\nabla = \left\{ v \in H^1_0(\Omega) : |\nabla v| \leq 1 \right\}$ such that, for $\beta$ a positive constant,

$$\int_\Omega \nabla u \cdot \nabla (v - u) \geq \beta \int_\Omega (v - u) \quad \forall\, v \in \mathbb{K}^\nabla,$$

is known to be equivalent to the same variational inequality with the convex set $\mathbb{K}^\nabla$ replaced by

$$\mathbb{K} = \left\{ v \in H^1_0(\Omega) : -d(\cdot, \partial\Omega) \leq v \leq d(\cdot, \partial\Omega) \right\},$$

where $d$ denotes the euclidean distance. For details about the above results see [2–4], among others.

These remarks can easily be extended to the evolutionary case (for which the existence of solution is not so obvious). The solution $u$, belonging to a suitable space, such that $u(t) \in \mathbb{K}^\nabla$, $u(0) = u_0 \in \mathbb{K}^\nabla$ and

$$\int_\Omega \partial_t u(t)(v - u(t)) + \int_\Omega \nabla u(t) \cdot \nabla (v - u(t)) \geq \beta(t) \int_\Omega (v - u(t)) \quad \forall\, v \in \mathbb{K}^\nabla, \text{ for a.e. } t \in (0, T),$$

is also the solution of the same variational inequality with convex set $\mathbb{K}$. Here, $\partial_t$ denotes the partial derivative with respect to the variable $t$ and $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$ is the spatial gradient.

There exists a commitment between the two approaches. It is easier to prove existence of solution of the two obstacles problem but, on the other hand, the solution of the problem with gradient constraint is immediately more regular, more

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precisely, it belongs to $L^\infty\big(0,T;W^{1,\infty}(\Omega)\big)$. So, it may be relevant to prove that the same function solves the two problems in more general situations.

Several physical problems are modelled using variational formulations with constraints on the first derivatives of the solutions. Examples can be found, for instance, in the papers [12,11,7,13].

An interesting model consists of a system of an electromagnetic variational inequality, where the curl of the magnetic field is constrained by a function of the temperature, which is solution of the heat equation with source term depending on the magnetic field. Considering $\Omega$ a bounded open subset of $\mathbb{R}^3$ and denoting by $Q$ the cylinder $\Omega\times(0,T)$ and by $\Sigma$ the lateral surface $\partial\Omega\times(0,T)$, the authors solved, in [1], a simplified version of this model, by considering a longitudinal geometry, more precisely, assuming that the magnetic field $\boldsymbol{h}=(0,0,u)$. So, $\nabla\times\boldsymbol{h}=(\partial_{x_2}u,-\partial_{x_1}u,0)$, and we arrive at the following coupled problem with gradient constraint (for details see [1]): to find $(u,\theta)$ belonging to a convenient space such that

$$\begin{cases} u(t)\in\mathbb{K}^{\nabla}_{F(\theta(t))} \quad \text{for a.e. } t\in(0,T), \qquad u(0)=u_0\in\mathbb{K}^{\nabla}_{F(\theta_0)}\\ \displaystyle\int_\Omega \partial_t u(t)(v-u(t))+\int_\Omega\nabla u(t)\cdot\nabla(v-u(t))\ \geq\ \int_Q f(t)(v-u(t)) \quad \forall\, v\in\mathbb{K}^{\nabla}_{F(\theta(t))},\ \text{for a.e. } t\in(0,T);\end{cases} \tag{1}$$

$$\begin{cases}\partial_t\theta-\triangle\theta=g(u) & \text{in } Q\\ \theta=0 & \text{on } \Sigma\\ \theta(0)=\theta_0 & \text{on } \Omega,\end{cases}\tag{2}$$

where $f:Q\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $F:\mathbb{R}\to\mathbb{R}^+$ are given functions and

$$\mathbb{K}^{\nabla}_{F(\theta(t))}=\left\{v\in H^1_0(\Omega):|\nabla v|\le F(\theta(t)) \text{ a.e. in } \Omega\right\}.\tag{3}$$

A natural way to solve evolutionary variational inequalities consists of approximating them by a family of penalized equations. As far as the authors know, the only type of explicit regular penalization of a variational inequality with gradient constraint is performed by a family of quasilinear parabolic equations. As a consequence of this penalization and of the coupling, *a priori* estimate of the time derivative of the solution of the variational inequality is obtained only in $\big(L^\infty(0,T;W^1_{\infty,0}(\Omega))\big)'$ and the existence result for the coupled problem treated in [1] does not correspond to a strong variational formulation. We remark that the variational inequality is, in fact, quasi-variational, as the convex set depends on $\theta$, solution of the heat equation, which depends on $u$. To treat this kind of problems, there exist other possible approaches, for instance semigroup theory, as in the one-dimensional quasi-variational evolutionary problem in [5] or time dependent subdifferential theory, as in [6].

At least for special classes of data, we are able to prove that a solution of a two obstacles coupled problem also solves the gradient constraint coupled problem, improving the "regularity" result obtained in [1] for the time derivative of this solution and obtaining existence for the strong variational formulation.

Although the main motivation for this paper is the one indicated above, it is worthwhile to prove existence of solution of the following more general two obstacles coupled system: given functions $f:Q\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, operators $A$, $B$ with domain conveniently defined and the convex set

$$\mathbb{K}^{B\theta(t)}_{A\theta(t)}=\Big\{v\in H^1_0(\Omega):A\theta(t)\le v\le B\theta(t) \text{ a.e. in } \Omega\Big\},$$

to find $(u,\theta)$ in a convenient space, solving the system (4)–(2), where

$$\begin{cases} u(t)\in\mathbb{K}^{B\theta(t)}_{A\theta(t)} \quad \text{for a.e. } t\in(0,T), \qquad u(0)=u_0\in\mathbb{K}^{B\theta_0}_{A\theta_0}\\ \displaystyle\int_\Omega \partial_t u(t)(v-u(t))+\int_\Omega\nabla u(t)\cdot\nabla(v-u(t))\ \geq\ \int_Q f(t)(v-u(t)) \quad \forall\, v\in\mathbb{K}^{B\theta(t)}_{A\theta(t)},\ \text{for a.e. } t\in(0,T).\end{cases}\tag{4}$$

In Section 2 we present two examples of special choices for the operators $A$ and $B$, a theorem of existence and regularity of the solutions of problem (4)–(2) and a theorem of existence of regular solutions of problem (1)–(2), with additional assumptions on the data. The proof of this last theorem will use the results obtained for the two obstacles system.

Section 3 is concerned with the proof of existence of solution of a coupled penalized problem that approximates the two obstacles coupled problem, based on obtaining *a priori* estimates and applying Schauder fixed point theorem. We also prove the first theorem of Section 2, more precisely, existence of solution of problem (4)–(2), as well as the existence of solution for the operators $A$ and $B$ chosen in the first example.

In Section 4 we prove existence of solution for the problem with operators $A$ and $B$ given by the second example. Besides, we prove that the solutions of this last problem solve the gradient constraint problem (1)–(2), if suitable assumptions are imposed on the data.

Relevant open questions for the coupled two obstacles problem are the uniqueness of solution as well as its asymptotic behaviour in time.

In this paper we use, for instance, the notation $W^{2,1}_p(Q)$ for the Sobolev space $W^1_p\big(0,T;L^p(\Omega)\big)\cap L^p\big(0,T;W^2_p(\Omega)\big)$ and, for consistency of notations, we use $W^1_p(\Omega)$ and $W^1_{p,0}(\Omega)$ instead of $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$, respectively.