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On strongly indefinite systems involving the fractional Laplacian

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ABSTRACT

This paper is devoted to study strongly indefinite systems involving the fractional Laplacian on bounded domains. Explicitly, we obtain existence and non-existence results, *a priori* estimates of Gidas–Spruck type, and a symmetry result. In addition, we give a different proof for the *a priori* estimate for nonlinear elliptic problems with the fractional Laplacian obtained in Cabré and Tan (2010) and Tan (2013).

1. Introduction

In this paper we shall study the following nonlinear problem

$$\begin{cases} \mathcal{A}_s u = v^p & \text{in } \Omega, \\ \mathcal{A}_s v = u^q & \text{in } \Omega, \\ u > 0, \qquad v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $p > 1$, $q > 1$, Ω is a smooth bounded domain of $\mathbb{R}^n$ and $\mathcal{A}_s$ denotes the fractional Laplace operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary values on $\partial\Omega$, defined in terms of the spectra of the Dirichlet Laplacian $-\Delta$ on Ω.

The problem (1.1) with $\Omega = R^n$ has been studied by many authors (see e.g. [12,13,35]). The problem was handled as an integro-differential system by inverting the operator $(-\Delta)^s$ to $(-\Delta)^{-s}$. This interpretation is particularly convenient in the case $\Omega = \mathbb{R}^n$.

Recently, Caffarelli and Silvestre [10] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^n$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. This observation made a significant influence on the study of related nonlocal problems. A similar extension

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was devised by Cabré and Tan [9] and Capella, Dávila, Dupaigne, and Sire [11] (see Brändle, Colorado, de Pablo, and Sánchez [7] and Tan [33] also). Based on this local interpretation, we shall derive many important properties of the solutions to the nonlocal system (1.1).

The fractional Laplacian defined on $\mathbb{R}^n$ appears in diverse areas including physics, biological modeling and mathematical finances and partial differential equations involving the fractional Laplacian have attracted the attention of many researchers. Among them, many authors studied nonlinear problems of the form $\mathcal{A}_s u = f(u)$ with a certain function $f : \mathbb{R}^n \to \mathbb{R}$, both in bounded domains and the whole space $\mathbb{R}^n$. Here we shall state some of the works on bounded domains. When $s = \frac{1}{2}$, Cabré and Tan [9] established the existence of positive solutions for equations having nonlinearities with the subcritical growth, their regularity, the symmetric property, and a priori estimates of the Gidas–Spruck type by employing a blow-up argument along with a Liouville type result for the square root of the Laplacian in the half-space. Brändle, Colorado, de Pablo, and Sánchez [7] dealt with a subcritical concave–convex problem. For $f(u) = u^q$ with the critical and supercritical exponents $q \geq \frac{n+2s}{n-2s}$, the nonexistence of solutions was proved in [7,32,33] in which the authors devised and used the Pohozaev type identities. The Brezis–Nirenberg type problem was studied in [4] for $0 < s < 1$ and [32] for $s = 1/2$ independently. The author and Seok [15] obtained the existence of infinitely many solutions for the Brezis–Nirenberg problem. Also, the author with Kim and Lee [14] studied the asymptotic behavior of solutions to the slightly sub-critical problem and the Brezis–Nirenberg type problem for $0 < s < 1$.

When $s = 1$ the nonlinear problem (1.1) corresponds to the well-known Lane–Emden system,

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ u > 0, \quad v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This system is a fundamental form among strongly coupled nonlinear systems and so it has received a lot of interest from many authors. Let us mention some of the fundamental works among them. For the nonlinear system (1.2) we say that $(p,q) \in (1,\infty) \times (1,\infty)$ is subcritical if $1/(p+1) + 1/(q+1) > (n-2)/n$, critical if $1/(p+1)+1/(q+1) = (n-2)/n$, and supercritical if $1/(p+1)+1/(q+1) < (n-2)/n$. The existence of nontrivial solution to (1.2) for subcritical cases was obtained by Hulshof–Vorst [23] and by Figueiredo–Felmer [19] independently. We also mention a previous result of Clément–Figueiredo–Mitidieri [16] where the authors obtained an existence result with nonlinearity having subcritical behavior near zero and linear behavior infinity. Later, Clément and Vorst [17] introduced a functional framework which can generalize the system and a multiparameter Hamiltonian system originating in equations of electrostatics. See also [18] where the authors obtain the existence result with general nonlinearities satisfying a suitable growth condition. On the other hand, Mitidieri [28] obtained the Pohozaev type identity for the nonlinear system (1.2) which implies the nonexistence of positive solutions to (1.2) for the critical and supercritical cases when the domain is starshaped (see also [29] for a generalized version). Hulshof–Mitidieri–Vorst [22] found nontrivial solutions to (1.2) with linear perturbation when (p,q) is critical, which can be seen as the Brezis–Nirenberg type problem for the system (1.2). Generally, nonlinear systems come from mathematical modeling such as Gierer–Meinhardt type system and solitary waves of coupled Schrödinger systems (see [2,3,5,25,27,31]). We refer to the book [30] for a survey of this topic.

Before studying the problem (1.1) we shall establish a different proof for *a priori* estimate for solutions to the problem

$$\begin{cases} \mathcal{A}_s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$