Nonlinear Analysis 120 (2015) 186–201

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Hölder weak sharp minimizers and Hölder tilt-stability[☆]

Xi Yin Zheng[a,*], Kung-Fu Ng[b]

[a] Department of Mathematics, Yunnan University, Kunming 650091, PR China
[b] Department of Mathematics (and IMS), Chinese University of Hong Kong, Hong Kong, China

ARTICLE INFO

Article history:
Received 20 September 2014
Accepted 28 February 2015
Communicated by S. Carl

MSC:
90C31
49K40
49J52

Keywords:
Hölder weak sharp minimizer
Stable Hölder weak sharp minimizer
Hölder tilt-stable minimizer
Hölder metric regularity

ABSTRACT

In this paper, we introduce and study the notions of Hölder weak sharp minimizers, stable Hölder weak sharp minimizers and Hölder tilt-stable weak minimizers for a proper lower semicontinuous function f on a Banach space. In terms of the Hölder metric subregularity/regularity of ∂f, we consider optimality conditions for Hölder weak sharp minimizers and stable Hölder weak sharp minimizers. We prove that $\bar{x}$ is a stable Hölder weak sharp minimizer (resp. a Hölder tilt-stable weak minimizer) of f if and only if it is a stable Hölder sharp minimizer (resp. a Hölder tilt-stable minimizer) of f.

1. Introduction

Let X be a Banach space, and we consider a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ (with the effective domain and the Clarke–Rockafellar subdifferential denoted by $\mathrm{dom}(f)$ and ∂f; see the next section for definitions and notations). Recall (cf. [3,7]) that $\bar{x} \in \mathrm{dom}(f)$ is a sharp minimizer (or strong local minimizer) of f if there exist positive constants κ and δ such that

$$\kappa\|x - \bar{x}\| \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta), \tag{1.1}$$

where $B_X(\bar{x}, \delta)$ denotes the open ball of X with center $\bar{x}$ and radius δ (and $B_X[\bar{x}, \delta]$ will be used to denote the corresponding closed ball). The notion of sharp minimizers has been recognized to be useful in convergence analysis of algorithms in optimization. However, the sharp minimizer notion in the sense of (1.1) is a rather strong condition: for example, it can be shown easily that a smooth function does not have any sharp minimizer. Replacing $\|x - \bar{x}\|$ in (1.1) by $\|x - \bar{x}\|^q$ with some constant $q > 1$, one can consider the following

☆ This research was supported by the National Natural Science Foundation of P. R. China (Grant No. 11371312 and 11261067), Earmarked Grants (GRF) from the Research Grant Council of Hong Kong (Project nos. CUHK 402612 and 14304014) and IRTSTY.

* Corresponding author.
E-mail addresses: xyzheng@ynu.edu.cn (X.Y. Zheng), kfng@math.cuhk.edu.hk (K.-F. Ng).

http://dx.doi.org/10.1016/j.na.2015.02.020

weaker notion of a q-order (sharp) minimizer of f: there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa\|x - \bar{x}\|^q \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta). \tag{1.2}$$

In the case of $q = 2$, the 2-order (sharp) minimizer notion in the sense of (1.2) is well-known and has played an important role in perturbation theory and convergence analysis in optimization. Recently, many authors studied the stable 2-order minimizers when the function f undergoes small linear perturbations by considering the functions

$$f_{u^*} := f - u^* \tag{1.3}$$

with u^* in X^* (cf. [2,5,6,12,15,17,16,18,20,21]). Replacing 2 by general q in $(1, +\infty)$, Zheng and Ng [28] further introduced the stable Hölder sharp minimizer: a point $\bar{x} \in \mathrm{dom}(f)$ is said to be a stable q-order (sharp) minimizer of f if there exist $\delta, r, \kappa \in (0, +\infty)$ such that for each $u^* \in B_{X^*}(0, \delta)$ there exists $x_{u^*} \in B_X(\bar{x}, r)$, with $x_0 = \bar{x}$, satisfying the following property:

$$\kappa\|x - x_{u^*}\|^q \leq f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X(\bar{x}, r). \tag{1.4}$$

Motivated by the tilt-stability of Poliquin and Rockafellar (see [21]), Zheng and Ng [28] also introduced the following notion: $\bar{x}$ is said to be a tilt-stable p-order minimizer of f (or say that $\bar{x}$ gives a tilt-stable p-order minimum of f) with $p \in (0, +\infty)$ if there exist $r, \delta, L \in (0, \ \infty)$ and $M : B_{X^*}(0, \delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that

$$f_{u^*}(M(u^*)) = \min_{x \in B_X[\bar{x}, r]} f_{u^*}(x) \quad \forall u^* \in B_{X^*}(0, \delta) \tag{1.5}$$

(where f_{u^*} is as in (1.3)) and

$$\|M(x^*) - M(u^*)\| \leq L\|x^* - u^*\|^p \quad \forall x^*, u^* \in B_{X^*}(0, \delta). \tag{1.6}$$

Significant advances have been made regarding the stable minimizers and the tilt-stable sharp minimizers for the case when $q = 2$ and $p = 1$ (cf. [1,2,5,6,12,15,16,18,20,21]). In particular, under the assumption that f is a proper lower semicontinuous function on a Hilbert space and $\bar{x}$ is a local minimizer of f such that f is subdifferentially continuous and proximally regular at $(\bar{x}, 0)$, the following statements are known to be equivalent:

(i) $0 \in \partial f(\bar{x})$ and the generalized second order subdifferential $\partial^2 f(\bar{x}, 0)$ (whose graph is the Mordukhovich normal cone of the graph of ∂f to $(\bar{x}, 0)$) is positively definite.
(ii) the subdifferential mapping ∂f is strongly metrically regular at $\bar{x}$ for 0.
(iii) $\bar{x}$ is a stable 2-order (sharp) minimizer of f.
(iv) $\bar{x}$ is a tilt-stable (1-order) minimizer of f.

Note that the above (i) has no counterpart in the general case when q is any number in $(1, +\infty) \setminus \{2\}$, majorly due to the fact that we do not have a satisfactory notion/theory for the corresponding higher order subdifferentials (especially no 'fractional-order' subdifferentials has been considered). Zheng and Ng [28] extended the mutual equivalences of (ii), (iii) and (iv) to the general case of $q \in (1, +\infty)$.

The q-order (sharp) minimizer in the sense of (1.2) is sufficiently strong to ensure that $\arg\min_{x \in B_X(\bar{x}, \delta)} f = \{\bar{x}\}$ is a singleton. In many cases, it is desirable to consider minimizers which are not necessarily isolated ones such as the weak sharp minimizers considered in the seminal paper by Ferris [7]: $\bar{x}$ is called a (local) weak sharp minimizer of f if there exist $r, \kappa \in (0, \ \infty)$ such that $f(\bar{x}) = \inf_{x \in B_X[\bar{x}, r]} f(x)$ and

$$\kappa d(x, S(f, \bar{x}, r)) \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, r), \tag{1.7}$$

where

$$S(f, \bar{x}, r) := \left\{x \in B_X[\bar{x}, r] : \ f(x) = \inf_{u \in B_X[\bar{x}, r]} f(u)\right\}.$$