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Zero dielectric constant limit of the full Magnet-Hydro-Dynamics system

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ABSTRACT

In this paper, we study the zero dielectric constant limit to the full Magnet-Hydro-Dynamics system (or more precisely the Maxwell–Navier–Stokes system). For well prepared initial data, the convergences of solutions of the full MHD system towards to the solutions of the classical 2D parabolic MHD system are justified rigorously by adapting the elaborate energy method as the dielectric constant tends to zero. Furthermore, the corresponding convergence rates are also obtained.

1. Introduction

The main objective of this paper is to study the zero dielectric constant limit of the full Magnet-Hydro-Dynamics system (the Maxwell–Navier–Stokes system) which is a couple system consisting of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The coupling comes from the Lorentz force in the fluid equation and the electric current in the Maxwell equations. The (scaled) Maxwell–Navier–Stokes system in two-dimensional torus has the following form [16,7]

$$\partial_t u^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon - \mu\Delta u^\epsilon + \nabla p^\epsilon = j^\epsilon \times B^\epsilon, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.1}$$

$$\epsilon\partial_t E^\epsilon - \operatorname{curl} B^\epsilon = -j^\epsilon, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.2}$$

$$\partial_t B^\epsilon + \operatorname{curl} E^\epsilon = 0, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.3}$$

$$\operatorname{div} u^\epsilon = \operatorname{div} B^\epsilon = 0, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.4}$$

$$j^\epsilon = \sigma(E^\epsilon + u^\epsilon \times B^\epsilon), \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.5}$$

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with the following initial condition

$$u^\epsilon(t=0) = u_0^\epsilon, \qquad E^\epsilon(t=0) = E_0^\epsilon, \qquad B^\epsilon(t=0) = B_0^\epsilon. \tag{1.6}$$

Here, u^ϵ is the velocity of the fluid, E^ϵ is the electric field, B^ϵ is the magnetic field. j^ϵ is the electric current which is given by Ohm's law. The force term $j^\epsilon \times B^\epsilon$ in the Navier–Stokes equations comes from Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. p^ϵ is the scalar pressure which can be recovered from u^ϵ and $j^\epsilon \times B^\epsilon$ via an explicit Calderon–Zygmund type operator [2], μ is the viscosity, σ is the electric conductivity and ϵ is the dielectric constant. For simplicity, we will take $\mu = \sigma = 1$. Eq. (1.1) is the Navier–Stokes equation for incompressible flows with a Lorentz force term. Eq. (1.2) is the Ampère–Maxwell equation for an electric field E^ϵ and Eq. (1.3) is the Faraday's law. Eq. (1.4) is the divergence free condition for u^ϵ and B^ϵ. Since the divergence-free condition of the magnetic field is conserved, $\mathrm{div} B^\epsilon = 0$ in (1.4) is not necessary in general if we assume the divergence-free condition for the initial data of the magnetic field in $\mathbb{T}^2$. Eq. (1.5) is the Ohm's law which states that the electric current is proportional to the electric field measured in a frame moving with the local velocity of the conductor. This explains the extra term $u^\epsilon \times B^\epsilon$. For a detailed physical introduction to the Magnet-Hydro-Dynamics system, we refer to Imai [8], Biskamp [1] and Davidson [4].

Note that in the two-dimensional case, the vector functions u^ϵ, E^ϵ and B^ϵ are defined on $\mathbb{T}^2$ and take their values in $\mathbb{R}^3$. This justifies the use of the cross product $u^\epsilon \times B^\epsilon$ and $j^\epsilon \times B^\epsilon$. In this case, the operator ∇ is given

$$\nabla = (\partial_{x_1}, \partial_{x_2}, 0)^T.$$

Thus

$$\mathrm{div} u^\epsilon = \frac{\partial u_1^\epsilon}{\partial_{x_1}} + \frac{\partial u_2^\epsilon}{\partial_{x_2}}, \qquad \nabla p^\epsilon = \left(\frac{\partial p^\epsilon}{\partial_{x_1}}, \frac{\partial p^\epsilon}{\partial_{x_2}}, 0\right)^T,$$

and

$$\mathrm{curl}\, G^\epsilon = \left(\frac{\partial G_3^\epsilon}{\partial_{x_2}}, -\frac{\partial G_3^\epsilon}{\partial_{x_1}}, \frac{\partial G_2^\epsilon}{\partial_{x_1}} - \frac{\partial G_1^\epsilon}{\partial_{x_2}}\right)^T.$$

The Maxwell–Navier–Stokes system (1.1)–(1.5) has been studied by many authors. Recently, for initial data $(u_0^\epsilon, E_0^\epsilon, B_0^\epsilon) \in L^2(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 0$, Masmoudi in [16] proved the existence and uniqueness of global strong solutions. For the initial data $(u_0^\epsilon, E_0^\epsilon, B_0^\epsilon) \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3)$ in three-dimensional case and $(u_0^\epsilon, E_0^\epsilon, B_0^\epsilon) \in \dot{B}^0_{2,1}(\mathbb{R}^3) \times L^2_{log}(\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3)$ in the bidimensional case, Ibrahim and Keraani in [6] built up the strong solutions. More recently, Ibrahim and Yoneda in [7] constructed local-in-time solution for non-decaying initial data on the torus $\mathbb{T}^3$ and showed the loss of smoothness of solutions. However, for the initial data lying in $L^2(\mathbb{R}^2)$, the global finite energy weak solution (Leray-type solution) to system (1.1)–(1.5) remains an interesting open problem in both dimensions $d = 2, 3$.

The purpose of this paper is to investigate the singular limit of the problem (1.1)–(1.5) in the so-called zero dielectric regime. Formally, taking the dielectric constant $\epsilon = 0$ in (1.2), we can have that $\mathrm{curl}\, B^0 = j^0$. Thanks to (1.5), we can replace the electric field E^0 by

$$E^0 = \mathrm{curl}\, B^0 - u^0 \times B^0 \tag{1.7}$$

in (1.1) and (1.3), and finally obtain that

$$\partial_t u^0 + (u^0 \cdot \nabla) u^0 - \Delta u^0 + \nabla p^0 = \mathrm{curl}\, B^0 \times B^0, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.8}$$

$$\partial_t B^0 - \mathrm{curl}\,\mathrm{curl}\, B^0 + \mathrm{curl}\,(u^0 \times B^0) = 0, \quad \text{in } \mathbb{T}^2 \times (0,T), \tag{1.9}$$

$$\mathrm{div} u^0 = \mathrm{div} B^0 = 0, \quad \text{in } \mathbb{T}^2 \times (0,T). \tag{1.10}$$