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Boundary partial regularity for solutions of quasilinear parabolic systems with non smooth in time principal matrix

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ABSTRACT

Partial regularity for solutions of quasilinear parabolic systems with non smooth in time principal matrix in the interior of space–time cylinder Q was proved in Arkhipova et al. (2014). In this paper we extend the results up to the parabolic boundary $\partial_p Q$. The coefficients of the system are assumed to be only bounded and measurable in the time variable. To prove the result, we apply the method of modified boundary A(t)-caloric approximation.

1. Introduction

In this paper we continue the study of partial regularity of weak solutions to quasilinear parabolic systems started in [2] by proving the partial regularity up to the parabolic boundary. We consider systems

$$v_t - \operatorname{div}(a(z,v)\nabla v) = g - \operatorname{div} G; \quad z \in Q, \tag{1}$$

where $z = (x,t) \in Q = \Omega \times (-T,0)$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $T > 0$ is an arbitrary fixed number. By v_t we denote the time derivative of a function $v : Q \to \mathbb{R}^N, N \geq 1$, and by $\nabla v = \left(\frac{\partial v^i}{\partial x_\alpha}\right)_{\alpha=1,\dots,n}^{i=1,\dots,N}$ its gradient with respect to the space variables. We assume that v satisfies the Cauchy–Dirichlet condition on the parabolic boundary $\partial_p Q$

$$v|_{\partial_p Q} = u^0, \tag{2}$$

where $\partial_p Q = (\partial\Omega \times (-T,0)) \cup \left(\overline{\Omega} \times \{-T\}\right)$ and u^0 is a fixed function. The assumptions on the matrix $a(z,v)$, the functions u^0, g and G will be described later.

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In [2] we considered system (1) and relaxed known conditions on the matrix $a(z,\eta)$ guaranteeing partial regularity of its weak solutions. Namely, we admitted the coefficients with integral continuity (VMO) with respect to the space variables and only bounded and measurable in the time variable t. Here we extend this result up to the parabolic boundary $\partial_p Q$ for weak solutions of problem (1), (2).

Solvability and regularity of solutions to various classes of nonlinear *scalar* parabolic equations and *linear* systems of equations with principal matrix non smooth in time were studied earlier (see [11,12,7] and references therein). Here we continue to develop the A-caloric approximation method (see [8,9,3,4]) and prove so-called A(t)-caloric approximation lemma for parabolic problems under the Dirichlet boundary condition on the flat part of a model parabolic cylinder (Lemma 5 below).

Note that even if the coefficients of a quasilinear system depend on η only and are real analytic on a neighborhood of the set where solution v takes its values, for dimensions $n \geq 3, N \geq 3$ the system (1) can have a solution which develops a singularity in the interior of Q (see [16,17]). At the same time, the counter-example [10] shows that there exists a weak solution of an elliptic quasilinear system under trivial Dirichlet boundary condition which admits singular points at the flat part of the boundary. Thus, even for very smooth coefficients the *partial* regularity of solutions to parabolic systems is the limit we can arrive to.

In what follows we assume that the coefficients $a(z,\eta) = \left(a_{ij}^{\alpha\beta}(z,\eta)\right)_{i,j=1,\ldots,N}^{\alpha,\beta=1,\ldots,n}$ are Carathéodory functions and satisfy the following conditions:

H1. There are positive constants λ, Λ such that

$$(a(z,\eta)\xi,\xi) = a_{ij}^{\alpha\beta}(z,\eta)\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \quad \xi \in \mathbb{R}^{nN},$$
$$|(a(z,\eta)p,q)| \leq \Lambda|p||q|, \quad p,q \in \mathbb{R}^{nN},$$

for almost all $z \in Q$ and all $\eta \in \mathbb{R}^N$.

H2. For almost all $z \in Q$ and all $\eta, \nu \in \mathbb{R}^N$ it holds

$$|a(z,\eta) - a(z,\nu)| \leq \omega(|\eta-\nu|^2),$$

where $\omega(s)$ is a non decreasing, bounded and concave function on $[0,\infty)$ with $\lim_{s\to 0+}\omega(s) = \omega(0) = 0$.

H3. The coefficients $a_{ij}^{\alpha\beta}(.,t,\eta)$ belong to $VMO(\Omega)$ for almost all $t \in (-T,0)$ and every $\eta \in \mathbb{R}^N, i,j \leq N$, $\alpha,\beta \leq n$, and

$$q^2(r) := \sup_{z^0=(x^0,t^0)\in Q\cup\partial_p Q}\ \sup_{\rho\in(0,r),\eta\in\mathbb{R}^N} \fint_{\Lambda_\rho(t^0)} \left(\fint_{\Omega_\rho(x^0)} |a(y,t,\eta) - a_{\rho,x^0}(t,\eta)|^2 dy\right) dt \to 0$$

for $r \to 0+$.

Here and below

$$\Lambda_\rho(t^0) = (t^0-\rho^2, t^0+\rho^2)\cap(-T,0), B_\rho(x^0) = \{x\in\mathbb{R}^n; |x-x^0|<\rho\}, \qquad \Omega_\rho(x^0) = B_\rho(x^0)\cap\Omega,$$
$$a_{\rho,x^0}(t,\eta) = \fint_{\Omega_\rho(x^0)} a(y,t,\eta)\,dy.$$

Further we reduce the non homogeneous boundary and initial conditions to the homogeneous ones. Assume that the function u^0 in (2) can be extended from $\partial_p Q$ to Q so that

$$u^0 \in W^{1,2}(Q).$$