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Infinitely many solutions for a fractional Kirchhoff type problem via Fountain Theorem

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ABSTRACT

In this paper, we use the Fountain Theorem and the Dual Fountain Theorem to study the existence of infinitely many solutions for Kirchhoff type equations involving nonlocal integro-differential operators with homogeneous Dirichlet boundary conditions. A model for these operators is given by the fractional Laplacian of Kirchhoff type:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u(x) - \lambda u = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $(-\Delta)^s$ is the fractional Laplacian operator with $0 < s < 1$ and $2s < N$, λ is a real parameter, M is a continuous and positive function and f is a Carathéodory function satisfying the Ambrosetti–Rabinowitz type condition.

1. Introduction

In this paper we are interested in the following fractional Laplacian equations of Kirchhoff type

$$\begin{cases} -M\left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy\right)\mathcal{L}_K u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

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Here Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega, N > 2s, s \in (0,1)$, while $\mathcal{L}_K$ is the integro-differential operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)\, dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$

with the kernel $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ such that

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N), \quad \text{where } \gamma(x) = \min\{|x|^2, 1\}; \\ \text{there exists } k_0 > 0 \text{ such that } K(x) \geq k_0\, |x|^{-(N+2s)} \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\}\,. \end{cases} \tag{1.3}$$

A prototype for K is given by the singular kernel $K(x) = |x|^{-(N+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$.

When $K(x) = |x|^{-(N+2s)}, \lambda = 0$ and $s \to 1^-$, problem (1.1) becomes the elliptic equation of Kirchhoff type

$$-M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u) \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, u satisfies some boundary conditions, see for example [1,13,14,24] for more information about Eq. (1.4). Note that Eq. (1.4) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u), \tag{1.5}$$

where $M(t) = a + bt$ for all $t \geq 0$, here $a, b > 0$, see for instance [37,38,45] for recent results. It was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{\lambda} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = f(x,u)$$

for free vibrations of elastic strings, see [19]. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial tension. It is worth pointing out that problem (1.5) received much attention only after Lions [22] proposed an abstract framework to the problem. It was pointed out in [1] that Eq. (1.5) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems.

When $K(x) = |x|^{-(N+2s)}, M \equiv 1$ and $\lambda = 0$, problem (1.1) conduces to the fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.6}$$

In recent years, a great attention has been focused on the study of the fractional Laplacian equation (1.6), see for example [16,17,28,29,26,35,34,42,43]. The fractional and nonlocal operators of elliptic type arise in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see for example [2,8,7,25] and the references therein. In the context of fractional quantum mechanics, nonlinear fractional Schrödinger equation has been proposed by Laskin [20,21] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. The literature on fractional and nonlocal operators and on their applications is quite large,