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Phase plane analysis for radial solutions to supercritical quasilinear elliptic equations in a ball

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ABSTRACT

We consider the following problem

$$\begin{cases} \Delta_p u + \lambda u + f(u,r) = 0 \\ u > 0 \quad \text{in } B, \quad \text{and} \quad u = 0 \quad \text{on } \partial B \end{cases} \tag{0.1}$$

where B is the unitary ball in $\mathbb{R}^n$. Merle and Peletier considered the classical Laplace case $p = 2$, and proved the existence of a unique value λ_0^* for which a radial singular positive solution exists, assuming $f(u,r) = u^{q-1}$ and $q > 2^* := \frac{2n}{n-2}$. Then Dolbeault and Flores proved that, if $q > 2^*$ but q is smaller than the Joseph–Lundgren exponent σ^*, then there is an unbounded sequence of radial positive classical solutions for (0.1), which accumulate at $\lambda = \lambda_0^*$, again for $p = 2$.

We extend both Merle–Peletier and Dolbeault–Flores results to the p-Laplace setting with the technical restriction $1 < p \leq 2$, and to more general nonlinearities f, which may have more complicated dependence on u and may be spatially non-homogeneous. Then we reproduce the results also for similar bifurcation problems where the linear term λu is replaced by a superlinear and subcritical term of the form $\lambda r^\eta u|u|^{Q-2}$. Our analysis relies on a generalized Fowler transformation and profits of invariant manifold theory, and it allows to discuss radial nodal solutions too.

1. Introduction

In this paper we study radial solutions for equations of the following form:

$$\Delta u + \lambda u + f(u,r) = 0 \qquad u(x) = 0 \quad \text{for } |x| = 1 \tag{1.1}$$

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where $r = |x|$, $x \in \mathbb{R}^n$. Abusing the notation we denote by $u(r)$ the radial solution $u(x)$ where $|x| = r$, so in fact we discuss the following singular O.D.E.

$$u'' + \frac{n-1}{r}u' + \lambda u + f(u,r) = 0 \qquad u(1) = 0. \tag{1.2}$$

Our results apply also to the equation

$$u'' + \frac{n-1}{r}u' + \lambda r^{\eta} u|u|^{Q-2} + f(u,r) = 0; \qquad u(1) = 0, \tag{1.3}$$

and to the generalization of (1.2) and (1.3) to the p-Laplace case for $1 < p \le 2$, see (1.6), (1.7) in the introduction. We assume f odd in u, positive for $u > 0$, superlinear for u small and supercritical for u large. We say that a solution $u(r)$ is regular if it is well defined for $r = 0$ and $u(0) = d > 0$, that it is singular if $\lim_{r\to 0} u(r) = +\infty$.

We denote by $2_* = 2\frac{n-1}{n-2}$ the Serrin critical exponent (related to the continuity of the trace operator), and by $2^* = \frac{2n}{n-2}$ the Sobolev critical exponent. We need also a further critical value, i.e.

$$\sigma^* = \begin{cases} 2\dfrac{n-2\sqrt{n-1}-2}{n-2\sqrt{n-1}-4} & \text{if } n > 10 \\ \sigma^* = \infty & \text{if } n \le 10. \end{cases} \tag{1.4}$$

The number σ^* is the so called Joseph–Lundgren exponent, introduced in [18] and is related to the existence of an ordering for regular solutions, and therefore plays a key role in associated parabolic problems, see e.g. [25,14]: its meaning will be further clarified later on.

Let us focus first on positive solutions of (1.1). The interest in these problems started from (1.1) and $f(u) = u^{q-1}$; in this case all positive solutions have to be radial, see [13,26] and this is the main motivation to reduce to (1.2): roughly speaking this is a general fact for f which are decreasing in r or r-independent. Let $\wedge_1$ be the first eigenvalue of $-\Delta$ in the unit ball with Dirichlet data. Multiplying the equation by the first eigenfunction and integrating by parts, it can be easily proved that (1.1) admits no positive solution for $\lambda \ge \wedge_1$ and for any $q > 2$ (whenever the domain is bounded, even with no symmetries). For $2 < q < 2^*$ there is at most one positive solution of (1.2) when $\lambda < \wedge_1$. When $q = 2^*$, Brezis and Nirenberg in [3] showed that (1.2) is solvable for $\bar{\lambda} < \lambda < \wedge_1$, where $\bar{\lambda} = 0$ for $n \ge 4$ and $\bar{\lambda} = \wedge_1/4$ for $n = 3$. So, in terms of standard bifurcation theory we can say that the set of pairs $(\lambda, u(0))$, where u is positive and solves (1.2), is a curve $\mathcal{C}$ that stems from $(\wedge_1, 0)$: if $2 < q \le 2^*$ the curve goes left without turning points, and it blows up as $\lambda \searrow \bar{\lambda}$ if $q = 2^*$. The situation changes drastically for $q > 2^*$. In this case, from the Pohozaev identity it follows that there are no positive solutions for $\lambda \le 0$. Moreover Merle and Peletier in [20] proved that there is a unique value $\lambda = \lambda_0^*$ such that (1.2) admits a positive singular solution. In [4] using numerical computations Budd and Norbury showed that the solutions curve turns right and oscillates infinitely many times across the curve $\lambda = \lambda_0^*$. Such a result was proved rigorously for $q < \sigma^*$ using phase plane analysis. From their argument it can be easily inferred that for $q \ge \sigma^*$ the curve $\mathcal{C}$ crosses at most finitely many times the curve $\lambda = \lambda_0^*$. The results obtained in [7] for positive solutions have been recently reproved by Guo and Wei in [15] using PDE techniques and evaluating the Morse index of the solutions. In fact they also showed that for $n \ge 12$ and q large enough (larger than a further critical exponent which is not explicitly computed, but equal or larger than σ^*), $\mathcal{C}$ goes left from $(\wedge_1, 0)$ without turning point and it blows up as $\lambda \searrow \lambda_0^*$.

Our main purpose is to find assumptions on f which are sufficient to reproduce the pattern described in [7] in the $2^* < q < \sigma^*$ case. Namely we prove the existence of the following patterns for (1.2) and (1.3) as λ varies.

S For any $k \in \mathbb{N}$ there is λ_k^* such that (1.2) (or (1.3)) admits a unique singular solution $u(\downarrow, r)$ with exactly k (non degenerate) zeros for $r \in (0,1)$. In particular for $\lambda = \lambda_0^*$, $u(\downarrow, r)$ is a positive solution of (1.2) (or (1.3)).