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Effect of resonance on the existence of periodic solutions for strongly damped wave equation

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ABSTRACT

We are interested in the differential equation $\ddot{u}(t) = -Au(t) - cA\dot{u}(t) + \lambda u(t) + F(t, u(t))$, where $c > 0$ is a damping factor, A is a sectorial operator and F is a continuous map. We consider the situation where the equation is at resonance at infinity, which means that λ is an eigenvalue of A and F is a bounded map. We introduce new geometrical conditions for the nonlinearity F and use topological degree methods to find T-periodic solutions for this equation as fixed points of Poincaré operator.

1. Introduction

In this paper we are interested in the following strongly damped wave equation

$$\begin{cases} u_{tt} - c\Delta u_t = \Delta u + \lambda u + f(t, x, u), & t \geq 0, \ x \in \Omega \\ u(t, x) = 0, & t \geq 0, \ x \in \partial\Omega \end{cases} \tag{1.1}$$

where $c > 0$ are damping factors, λ is a real number and $f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous map on an open bounded set $\Omega \subset \mathbb{R}^n$, which is T-periodic in time.

The existence of periodic solutions for damped wave equations has been investigated by many authors in the last years. In particular a large part of these studies concerns the weakly damped wave equation

$$\begin{cases} u_{tt} - cu_t = \Delta u + \lambda u + f(t, x, u), & t \geq 0, \ x \in \Omega \\ u(t, x) = 0, & t \geq 0, \ x \in \partial\Omega \end{cases} \tag{1.2}$$

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where the Laplacian in the damping term appears in the zero fractional power. For instance the results obtained in the series of papers [1,18,23,21,22] provide the existence of T-periodic solutions for (1.2), in the case when $\Omega \subset \mathbb{R}^n$ is *a thin domain*, that is, a Cartesian product of an open bounded subset of $\mathbb{R}^{n-1}$ and a small open interval. In these papers the periodic solutions are obtained as fixed points of the Poincaré operator by topological degree methods. On the other hand, we refer the reader to [7] where a homotopy invariants method is used to study the existence of periodic solutions in the case where Ω is an open interval and the damping term u_t is additionally involved with a nonlinearity. See also [5,14,16,31,35] for the results where Ω is again an open interval with the difference that the zero Dirichlet boundary conditions in Eq. (1.2) are replaced by the periodic one.

From the point of view of the mathematics, physics and engineering it is of importance to consider Eq. (1.1) in the presence of *resonance at infinity*, which means that

$$\mathrm{Ker}\,(\lambda I - A_2) \neq \{0\} \quad \text{and} \quad f \text{ is a bounded map,}$$

where we define $A_2 u := -\Delta u$ for $u \in D(A_2) := H^2(\Omega) \cap H_0^1(\Omega)$. Here we refer the reader to [28,29] for an extensive discussion on the meaning of resonance in the periodic oscillations of suspension bridges. The existence of periodic solutions for Eq. (1.1) in the case of the resonance at infinity was considered in [6] under the assumption that the damping constant $c = 0$. There was proved that the equation admits a periodic solution provided the nonlinearity f satisfies so called Landesman–Lazer type conditions. Subsequently, these conditions and topological degree methods were used in [9] to obtain the existence of periodic solutions in the weakly damped case (1.2).

In this paper our aim is to study the existence of T-periodic solutions for *the strongly damped wave equation* (1.1) in the presence of the resonance at infinity, which seems to be not an explored problem so far. Throughout the paper we will consider the more general abstract differential equation

$$\ddot{u}(t) = -Au(t) - cA\dot{u}(t) + \lambda u(t) + F(t, u(t)), \quad t \in [0, +\infty) \tag{1.3}$$

where $c > 0$ is still a damping constant, λ is a real number, $A : X \supset D(A) \to X$ is a positive sectorial operator with compact resolvents on a Banach space X and $F : [0, +\infty) \times X^\alpha \to X$ is a continuous map, where $X^\alpha = D(A^\alpha)$ for $\alpha \in (0, 1)$, is a fractional space endowed with the graph norm. For more details on the construction and properties of the fractional spaces we refer the reader to [19,20,36].

After passing into the abstract framework, we will say that Eq. (1.3) is at *the resonance at infinity*, provided

$$\mathrm{Ker}\,(\lambda I - A) \neq \{0\} \quad \text{and} \quad F \text{ is a bounded map.}$$

The main difficulty lies in the fact that, in the presence of resonance, there are examples of the nonlinearity F such that Eq. (1.3) does not admit a periodic solution. This fact will be explained in Remark 4.1. To overcome this difficulty we address the naturally arising question which says:

what additional assumptions for the nonlinearity F should be made to prove that Eq. (1.3) *admits a T-periodic mild solution.* (1.4)

To explain our methods more precisely, observe that Eq. (1.3) can be written in the following form

$$\dot{w}(t) = -\mathbf{A}w(t) + \mathbf{F}(t, w(t)), \quad t > 0,$$

where $\mathbf{A} : \mathbf{E} \supset D(\mathbf{A}) \to \mathbf{E}$ is a linear operator on the space $\mathbf{E} := X^\alpha \times X$ given by

$$D(\mathbf{A}) := \{(x, y) \in \mathbf{E} \mid x + cy \in D(A)\}$$
$$\mathbf{A}(x, y) := (-y, A(x + cy) - \lambda x) \quad \text{for } (x, y) \in D(\mathbf{A}),$$