Nonlinear Analysis 125 (2015) 201–226

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Double grow-up and global convergence of solutions for a parabolic prescribed mean curvature problem

Hongjing Pan[a], Ruixiang Xing[b,*]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

[b] School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou 510275, China

ARTICLE INFO

Article history:
Received 28 December 2014
Accepted 27 May 2015
Communicated by S. Carl

MSC:
35J93
35K93
53C44
53A10

Keywords:
Infinite time gradient blow-up
Prescribed mean curvature equation
Mean curvature flow
Double grow-up
Quasilinear parabolic problem
Global convergence

ABSTRACT

The paper is concerned with a parabolic mean curvature type problem with a varying parameter λ. We study large time behavior of global solutions for different ranges of λ and initial data and present some new asymptotic results about global convergence and infinite time blow-up. In particular, it is shown that for suitable ranges of parameter λ and initial data, there exists a *double grow-up* phenomenon: the solution itself blows up at every interior point and its gradient blows up at the boundary of the domain as $t \to +\infty$. We also establish an interesting connection between global convergence and the non-classical solution of the associated stationary problem: if initial data are smaller than the non-classical solution, then the solutions must decay to zero in C^1 norm as $t \to \infty$.

1. Introduction

Consider the following initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^{2+\alpha}$. We are interested in asymptotic behavior of solutions of (1.1) for large time.

* Corresponding author.
E-mail addresses: panhj@m.scnu.edu.cn (H. Pan), xingrx@mail.sysu.edu.cn (R. Xing).

http://dx.doi.org/10.1016/j.na.2015.05.031
0362-546X/

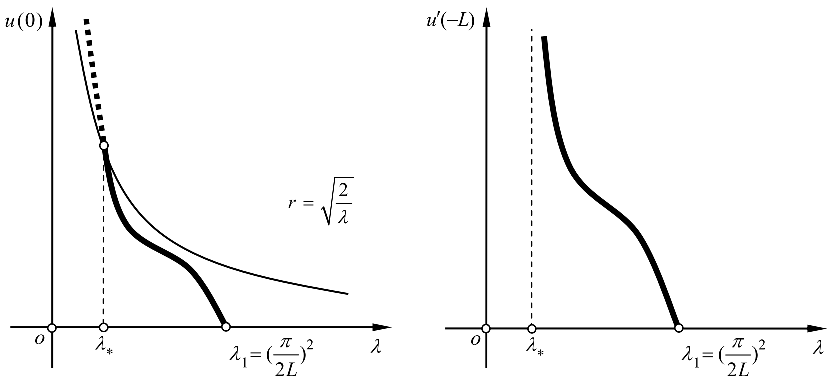

Fig. 1. Bifurcation diagrams for (1.3) in different parameter spaces.

The problem (1.1) is of both mathematical and physical interest. From the geometrical viewpoint, due to the mean curvature operator $-\mathrm{div}(\nabla u/\sqrt{1+|\nabla u|^2})$, this is a typical curvature evolutionary problem with force term λu. The stationary problem of (1.1) also comes from a model appearing in capillary surfaces theory and the negative steady state solutions describe pendent liquid drops of equilibrium state (see [7,26]). The problem (1.1) can also be viewed as nonlinear heat flow with unit heat capacity and conductivity $(1+|\nabla u|^2)^{-\frac{1}{2}}$; u represents for the temperature and λu is the heat source term (see [6]).

In recent years, the stationary problem of (1.1) has been well investigated; see [17,20,26] and the monograph [7]. More results about elliptic mean curvature type equations, see [2,19] and the references therein. However, only a little attention seems to be paid to the analysis of parabolic mean curvature equations (1.1), even for the case $N = 1$. We refer to [3,6,25].

In this paper, we focus on the one-dimensional case of (1.1), i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda u, & x \in (-L, L),\ t > 0, \\ u(-L,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = \varphi(x), & x \in (-L, L). \end{cases} \tag{1.2}$$

Throughout the paper, we assume that $\lambda > 0$, $\varphi(x) \in C^{2+\alpha}[-L, L]$, $\alpha \in (0, 1)$, and φ is compatible with the boundary condition $u = 0$. The prime $'$ denotes the derivative with respect to the spatial variable x.

The steady states of (1.2) satisfy

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda u, & x \in (-L, L), \\ u(-L) = u(L) = 0. \end{cases} \tag{1.3}$$

It is well known that there exists a number λ_*, with $0 < \lambda_* < \lambda_1 = (\frac{\pi}{2L})^2$, such that for $\lambda \in (\lambda_*, \lambda_1)$, (1.3) has exactly one nontrivial positive classical solution, while for $\lambda \in (0, \lambda_*] \cup [\lambda_1, +\infty)$, (1.3) has no nontrivial positive classical solutions (see e.g. [9,16,22]). Here, by a *classical* solution we mean a function $u \in C^2[-L, L]$ satisfying (1.3). Besides, it is also known that for $\lambda \in (0, \lambda_*]$, (1.3) has exactly one positive non-classical solution; while for $\lambda \in (\lambda_*, +\infty)$, (1.3) has no positive non-classical solutions (see [2,23]). These results are depicted by the bifurcation diagrams in Fig. 1, where $r = \max u = u(0)$, the thin curve $r = \sqrt{\frac{2}{\lambda}}$ in the left graph is the (gradient) blow-up curve (see [22] or [23]), the continuous thick line represents classical solutions, and the thick dashed line represents non-classical solutions. Here, the *non-classical* solution is defined in the sense of [2,18]. Precisely, a non-classical solution of (1.3) is a function $u : [-L, L] \to \mathbb{R}$, with $u \in C^2(-L, L)$, $u' \in C([-L, L], [-\infty, +\infty])$, and $u'(-L) = +\infty$ or $u'(L) = -\infty$, satisfying (1.3) on $(-L, L)$ (see Fig. 2 for