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Global nonexistence of positive initial energy solutions for a viscoelastic wave equation☆

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ABSTRACT

In this paper, we consider the nonlinear viscoelastic equation:

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-\tau) \triangle u(\tau) d\tau + |u_t|^{m-2} u_t = |u|^{p-2} u,$$
$$\text{in } \Omega \times [0,T],$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g, we prove the nonexistence of global solutions with positive initial energy.

1. Introduction

In this work, we are concerned with the following initial–boundary value problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u + \displaystyle\int_0^t g(t-\tau) \triangle u(\tau) d\tau + |u_t|^{m-2} u_t = |u|^{p-2} u, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega, m > 2, g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function, and

$$\begin{cases} 2 < p < \infty, \quad \text{if } n = 1, 2, \qquad 2 < p \leq \dfrac{2(n-1)}{n-2}, \quad \text{if } n \geq 3 \\ 0 < \rho < \infty, \quad \text{if } n = 1, 2, \qquad 2 < \rho \leq \dfrac{2}{n-2}, \quad \text{if } n \geq 3. \end{cases} \tag{1.2}$$

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The case of $\rho = 0$ in the absence the viscoelastic term ($g = 0$) has been extensively studied and results concerning existence and nonexistence have been established. In the further absence of damping mechanism $u_t|u_t|^{m-2}$, any solution with negative initial energy blows up infinite time [2,15]. In contrast, in the absence of the source $u|u|^{p-2}$, the damping term $u_t|u_t|^{m-2}$ assures global existence for arbitrary initial data, see [13,16]. The case of linear damping ($m = 2$) and nonlinear source was first considered by Levine [17,18], who introduced 'concavity arguments' and showed that solutions with negative initial energy blow up in finite time. Furthermore, the interaction between the nonlinear damping and the source terms was studied by Georgiev and Todorova [11], for a bounded domain with Dirichlet boundary conditions. For more related works, we refer the reader to [20,19,22,35,10,31,32].

In the case of $\rho = 0$ and in the presence of the viscoelastic term ($g \neq 0$), Messaoudi [23] considered the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau + au_t|u_t|^{m-2} = bu|u|^{p-2}, \quad \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega, p > 2,\ m \geq 1, a, b > 0$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function. He proved a blow up result for solution with negative initial energy if $p > m$, and a global result for $p \leq m$. This result was later improved by Messaoudi [24], to certain solutions with positive initial energy. A similar result was also obtained by Wu [37] using a different method. Further more, Song and Zhong [34] studied a nonlinear viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-\tau) \triangle u(\tau)d\tau - \triangle u_t = |u|^{p-2}u, \quad \text{in } \Omega \times [0,T], \end{cases} \tag{1.4}$$

where $2 < p < \frac{2n-2}{n-2}, \Omega \subset \mathbb{R}^n$, is a bounded domain with a smooth boundary $\partial\Omega$. The authors proved blow up for solutions with negative initial energy and also slightly positive energy using the "potential well" theory introduced by Payne and Sattinger [31]. Very recently, Song and Xue [33] proved a blow up result for solution of Eq. (1.4) with arbitrarily high initial energy. For the problem (1.3) in $\mathbb{R}^n$ and with $m = 2$, Kafini and Messaoudi [14] showed, under suitable conditions on g and initial data, that solution with negative energy blow up in finite time. In related work, Berrimi and Messaoudi [3] considered

$$u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau = u|u|^{p-2}, \quad \text{in } \Omega \times (0,\infty) \tag{1.5}$$

in a bounded domain and $p > 2$. They established a local existence result and showed, under weaker conditions than those in [7,9], that the local solution is global and decays uniformly if the initial data are small enough. In [1], an asymptotic stability and decay rates for solutions of the wave equation in star-shaped domains has were established by combination of memory effect and damping mechanism. In [6], an existence and decay result for viscoelastic problems with nonlinear boundary damping has been proved. For further work on existence and decay of solutions of a viscoelastic equation, we refer to [8,5,28,30,36].

The authors [4] studied the following problem ($\rho > 0$)

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-\tau)\triangle u(\tau)d\tau - \gamma\triangle u_t = 0, \quad \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.6}$$

and proved a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$. These results have been extended by Messaoudi and Tatar [27] to a situation where a source term is competing with the dissipation terms. Messaoudi and Tatar [29] studied the problem (1.6), in which the source term competes with only the viscoelastic dissipation induced by the memory term. They showed that there exists an appropriate set S (called a stable set) such that if the initial datum is in S then the solution continues to