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Second-order optimality conditions with arbitrary nondifferentiable function in scalar and vector optimization

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ABSTRACT

In this paper, we introduce a new second-order directional derivative and a second-order subdifferential of Hadamard type for an arbitrary nondifferentiable function. We derive second-order necessary and sufficient optimality conditions for a local and a global minimum and an isolated local minimum of second-order in unconstrained optimization. In particular, we obtain two results with strongly pseudoconvex functions. We also compare our conditions with the results of the recently published paper (Bednarik and Pastor, 2008) and a lot of other works, published in high level journals, and prove that they are particular cases of our necessary and sufficient ones. We prove that the necessary optimality conditions concern more functions than the conditions in terms of lower Dini directional derivative, even the optimality conditions with the last derivative can be applied to a function, which does not belong to some special class. At last, we apply our optimality criteria for scalar problems to derive necessary and sufficient optimality conditions in the cone-constrained vector optimization.

1. Introduction

In our opinion the main aim of nondifferentiable optimization is to extend some results to as more as possible general classes of functions. The task to obtain optimality conditions in unconstrained optimization is old. There are first- and second-order necessary and sufficient conditions which concern several classes of functions ($C^{1,1}$, C^1, locally Lipschitz, lower semicontinuous and so on) in terms of various generalized derivatives. For all of them we should check that the function belongs to some special class, which is not easy sometimes. There are a lot of second-order generalized directional derivatives, whose necessary and sufficient conditions for optimality have similar proofs (see, for example, Refs. [2–9,15,19–22,24,30–35]). Some of them are presented in the books [17,25,26]. This fact motivated us to find another derivative such that these conditions follow from the second-order ones in terms of it, the second-order necessary conditions and the sufficient ones in unconstrained optimization are satisfied for arbitrary nondifferentiable function and the

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derivative coincides with the second-order Fréchet directional derivative in the case when the last one exists. They can be applied in nonlinear programming, for example, for solving the problem with penalty functions or reduce the problem to convex composite.

In Section 2 of this paper, we introduce a new second-order generalized directional derivative with the mentioned above properties. In Section 3, we obtain necessary and sufficient conditions for a local minimum and isolated local minimum of a function in terms of this derivative. In the conditions, we suppose that the function is arbitrary proper extended. In Section 4, we prove necessary and sufficient first-order conditions for a given point to be an isolated minimizer of order two of a strongly pseudoconvex function. Our generalized derivatives have the advantage that the proofs of the optimality conditions are simple. On the other hand, they are satisfied for arbitrary function, not necessarily with locally Lipschitz gradient, or continuously differentiable, or locally Lipschitz, or continuous, even not necessarily semicontinuous. We also compare our necessary and sufficient conditions with the respective ones in Refs. [2–9,15,19–22,30,31,33–35] in Section 6. We prove that the conditions in all these works are simple consequences of our necessary and sufficient conditions. On the other hand, the proofs given there are not so short sometimes. For example, the main result in the recently published paper [4] is to extend the conditions for an isolated local minimum in unconstrained optimization to l-stable functions. This is a class of functions, whose lower Dini directional derivatives satisfy a property, which is analogous to Lipschitz one. They include all $C^{1,1}$ functions. We prove that the main Theorem 6 in this paper follows from Theorem 3.2 when the function is l-stable at the candidate for minimizer and continuous near it. Therefore, it is not necessary to guess and check if the function is l-stable. We also compare the necessary conditions in terms of Hadamard and Dini derivatives. We prove that our conditions are preferable. They concern more functions.

In Section 5, we obtain necessary and sufficient conditions for optimality in cone-constrained vector optimization. In particular, our results are satisfied for problems with inequality and equality constraints.

2. A new second-order directional derivative and subdifferential of Hadamard type

When one introduces higher-order directional derivatives in nonsmooth optimization and establishes in their terms optimality conditions the following demands are welcomed:

1. The conditions should work for arbitrary optimized function.
2. The conditions should be both necessary and sufficient.
3. The directional derivatives should be consistent with the classical ones for sufficiently smooth functions.

This idea was followed in Ref. [13], but not fulfilled with respect to the point 3.

We suppose that $\mathbb{E}$ is a real finite-dimensional Euclidean space. Denote by $\mathbb{R}$ the set of reals and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let us consider the following second-order directional derivative at the point $x \in \mathbb{E}$ in direction $u \in \mathbb{E}$ of a given function f, defined in the space $\mathbb{E}$, which was introduced in [13]:

$$f_G^{[2]}(x;u) := \liminf_{t\downarrow 0, u'\to u} 2t^{-2}[f(x+tu') - f_G^{[0]}(x;u) - tf^{[1]}(x;u)],$$

where $f_G^{[0]}(x;u) := \liminf_{t\downarrow 0, u'\to u} f(x+tu')$ and

$$f_G^{[1]}(x;u) := \liminf_{t\downarrow 0, u'\to u} t^{-1}[f(x+tu') - f_G^{[0]}(x;u)].$$

The optimality conditions for unconstrained problems were derived for arbitrary nondifferentiable function. Suppose that the function is twice Fréchet differentiable. Then $f_G^{[0]}(x;u) = f(x)$, $f_G^{[1]}(x;u) = \nabla f(x)(u)$ and

$$f_G^{[2]}(x;u) := \liminf_{t\downarrow 0, u'\to u} 2t^{-2}[f(x+tu') - f(x) - t\nabla f(x)(u)]. \tag{2.1}$$