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Solvability of the initial–boundary value problem of the Navier–Stokes equations with rough data[☆]

Tongkeun Chang[a], Bum Ja Jin[b,*]

[a] Department of Mathematics, Yonsei University, Seoul, 136-701, South Korea
[b] Department of Mathematics, Mokpo National University, Muan-gun 534-729, South Korea

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ABSTRACT

In this paper, we study the initial and boundary value problem of the Navier–Stokes equations in the half space. We prove the unique existence of weak solution $u \in L^q(\mathbb{R}^n_+ \times (0,T))$ with $\nabla u \in L^{\frac{q}{2}}_{loc}(\mathbb{R}^n_+ \times (0,T))$ for a short time interval when the initial data $h \in B_q^{-\frac{2}{q}}(\mathbb{R}^n_+)$ and the boundary data $g \in L^q(0,T; B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1})) + L^q(\mathbb{R}^{n-1}; B_q^{-\frac{1}{2q}}(0,T))$ with normal component $g_n \in L^q(0,T; \dot{B}_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))$, $n+2 < q < \infty$ are given.

1. Introduction

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$, $n \geq 2$ and $0 < T < \infty$. Let us consider the nonstationary Navier–Stokes equations

$$\begin{aligned} &u_t - \Delta u + \nabla p = f - \mathrm{div}\,(u \otimes u), \qquad \mathrm{div}\ u = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,T), \\ &u|_{t=0} = h, \qquad u|_{x_n=0} = g. \end{aligned} \tag{1.1}$$

There are abundant literatures for the study of the Navier–Stokes equations with homogeneous boundary data. See [1,6,28,34] and references therein for the half space problem. See also [1,9,14,16,17,21,23,22,24,29,31] and the references therein for the problems in other domains such as whole space, a bounded domain, or exterior domain.

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* Corresponding author.
E-mail addresses: chang7357@yonsei.ac.kr (T. Chang), bumjajin@hanmail.net (B.J. Jin).

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Over the past decade, the Navier–Stokes equations with the nonhomogeneous boundary data have been studied actively. See [8,3,2,32,41] and references therein for the half space problem. See also [3,2,11,10,12, 13,18–20,29] and the references therein for the problems in other domains such as whole space, a bounded domain, or exterior domain.

In [18–20,34], the solvabilities of bounded or exterior domain problem have been studied for a boundary data in anisotropic space $B_{q0}^{\alpha-\frac{1}{q},\frac{\alpha}{2}-\frac{1}{2q}}(\partial\Omega\times(0,T))$, $\alpha>\frac{1}{q}$ (with $q>\frac{n+2}{\alpha+1}$), where $g\in B_{q0}^{s,\frac{s}{2}}(S\times(0,T))$ means the zero extension of g to $S\times(-\infty,T)$ is in $B_q^{s,\frac{s}{2}}(S\times(-\infty,T))$. On the other hand, in [8,3,2,11, 10,12,13,32,41] rough boundary data have been considered. H. Amann [3] showed unique maximal solution $u\in L^r_{loc}(0,T^*,H_q^{\frac{1}{r}}(\Omega))$, $3<q<r<\infty,\frac{1}{r}+\frac{3}{r}\le 1$ for some maximal time T^* in any domain in $\mathbb{R}^3$ with nonempty compact smooth boundary when a nonzero initial data in $B_q^{-\frac{1}{r}}(\Omega)\cap L^q_\sigma(\Omega)$ and nonzero boundary data in $L^r_{loc}(\mathbb{R}_+;W_q^{-\frac{1}{q}+\frac{1}{r}}(\partial\Omega))$ are given. J.E. Lewis [32] showed a global in time existence of solution in $L^p(\mathbb{R}_+;L^q(\mathbb{R}^n_+))$ for small data $h\in L^{r_1}(\mathbb{R}^n_+)\cap L^{r_2}(\mathbb{R}^n_+)$ and $g\in L^d(\mathbb{R}_+;L^r(\mathbb{R}^n_+))$ with $r_1,r_2,p,q,r,d<\infty$, $r_1<n<r_2$, $\frac{n-1}{r}+\frac{2}{d}=1$, and $\frac{2}{q}+\frac{2}{p}=1$. K.A. Voss [41] showed the existence of a global in time self-similar solution for small data $h\in\dot B_{6,\infty}^{-\frac{1}{2}}(\mathbb{R}^3_+)\cap\dot B_{4,\infty}^{-\frac{1}{4}}(\mathbb{R}^3_+)$ and $t^{\frac{1}{3}}g(t)\in L^\infty(\mathbb{R}_+;L^3(\mathbb{R}^2))$ with $g_n=0$. M. Fernandes de Almeida and L.C.F. Ferreira [8] showed the existence of global in time solution in the framework of Morrey space for a small data $h\in\mathcal{M}_{p,n-p}(\mathbb{R}^n_+)$, $t^{\frac{1}{2}-\frac{p-1}{2r}}g\in BC(\mathbb{R}_+,\mathcal{M}_{r,n-p}(\mathbb{R}^{n-1}))$ and $t^{\frac{1}{2}-\frac{p}{2q}}g_n\in BC(\mathbb{R}_+,\mathcal{M}_{\frac{(p-1)q}{p},n-p}(\mathbb{R}^{n-1}))$, $2<p,q<\infty$, $1<r<\infty$.

In particular, R. Farwig, H. Kozono and H. Sohr [12] showed the local in time existence of a very weak solution $u\in L^s(0,T;L^q(\Omega))$ in an exterior domain when nonzero initial in $B_{q,s}^{-\frac{2}{s}}$ and nonzero boundary data in $L^s(0,T;W_q^{-\frac{1}{q}}(\partial\Omega))$ for $\frac{2}{s}+\frac{3}{q}=1,2<s<\infty,3<q<\infty$ are given. (Precisely speaking, in [12] a nonzero divergence is considered.)

In this paper, we show the unique existence of $u\in L^q(\mathbb{R}^n_+\times(0,T))$ with $\nabla u\in L^{\frac{q}{2}}(\mathbb{R}^n_+\times(0,T))$ for the Navier–Stokes equations (1.1) for a small time interval $(0,T)$ with the initial $h\in B_q^{-\frac{2}{q}}(\mathbb{R}^n_+)$ and the boundary data $g\in L^q(0,T;B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))+L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))$ with $g_n\in L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))$, $q>n+2$. Our result could be compared with the one in [12]. The case $q=r=5$ in [12] coincides with the case $q=5$ in our result, except the fact that our result covers larger class for g' (the tangential component of the boundary data) since $L^q(0,T;B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))+L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))\supsetneq L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))$.

The following is the main result of this paper.

Theorem 1.1. *Let $\infty>q>n+2$. Assume that $h\in B_q^{-\frac{2}{q}}(\mathbb{R}^n_+)$ with $\operatorname{div}h=0$, $g\in L^q(0,T;B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))+L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))$ with $g_n\in L^q(0,T;\dot B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))$. Then there is $T^*(0<T^*<T)$ so that the Navier–Stokes equations (1.1) have a unique weak solution $u\in L^q(\mathbb{R}^n_+\times(0,T^*))$ with $\nabla u\in L^{\frac{q}{2}}_{loc}(\mathbb{R}^n_+\times(0,T^*))$.*

The space $L^q(0,T;B_q^{-\frac{1}{q}}(\mathbb{R}^{n-1}))+L^q(\mathbb{R}^{n-1};B_q^{-\frac{1}{2q}}(0,T))$ coincides with anisotropic Besov space $B_q^{-\frac{1}{q},-\frac{1}{2q}}(\mathbb{R}^{n-1}\times\mathbb{R}_+)$ (see Section 2). Our result is optimal in the sense that the spaces for the initial and the boundary data cannot be enlarged for our solution class. Our arguments in this paper are based on the elementary estimates of the heat operator and the Laplace operator. The solution representation in Section 5.1 could be useful to study asymptotic behavior of the solution.

Before proving Theorem 1.1, we have studied the initial and boundary value problem of the Stokes equations in $\mathbb{R}^n_+\times(0,T)$ as follows:

$$\begin{aligned} &u_t-\Delta u+\nabla p=f, \qquad \operatorname{div}u=0, \quad \text{in } \mathbb{R}^n_+\times(0,T),\\ &u|_{t=0}=h, \qquad u|_{x_n=0}=g. \end{aligned} \tag{1.2}$$