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Bent rectangles as viscosity solutions over a circle

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ABSTRACT

We study the motion of the so-called bent rectangles by the singular weighted mean curvature. We are interested in the curves which can be rendered as graphs over a smooth one-dimensional reference manifold. We establish a sufficient condition for that. Once we deal with graphs we can have the tools of the viscosity theory available, like the Comparison Principle. With its help we establish uniqueness of variational solutions constructed by the authors (Giga et al., 2013). In addition, we establish a criterion for the mobility coefficient guaranteeing vertex preservation.

1. Introduction

An important part of a model of a single crystal growing in the atmosphere or from a solution is the Gibbs–Thomson law on a crystal surface, see [23,24,32,33],

$$\beta V = \kappa_\gamma + \sigma. \tag{1.1}$$

This equation relates the velocity of the advancing surface V to its weighted mean curvature κ_γ and the amount of matter σ, where $\beta > 0$ denotes the mobility depending on the orientation of the surface. The interpretation of σ depends upon the particular phenomenon we discuss, but in all cases we consider, σ depends upon the position on the plane and time. Moreover, it satisfies the structural assumption (2.1). The

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meaning of symbols used in (1.1) is explained in Section 2.1. We shall see then, that from the point of view of differential equations (1.1) is the weighted mean curvature flow with forcing.

In the full model the Gibbs–Thomson relation, (1.1), is coupled to the diffusion equation for supersaturation σ (temperature, pressure, etc.). The literature is abundant since this topic has been studied for a long time, a biased sample is: [1,10,12,27–30] and references therein. Any realistic modeling attempt should take into account, see [3]:

(1) lack of smoothness of the growing surface;
(2) lack of smoothness of the surface energy function appearing in the definition of the wmc κ_γ.

If we, in addition, expect a qualitative analysis of solutions, then there is probably no theory available.

We studied, in [18] (see also references therein), the evolution of bent rectangle by (1.1). Deformed rectangles are approximate cross sections of columnar ice crystals found in Nature, as this is seen from Nakaya diagram, see [26]. We established, in [18], the existence of variational solutions to (1.1) for bent rectangles (see the definition in Section 2), when σ is a given function conforming to the so-called Berg effect, see [9,19,25]. In the next section, we provide more details.

Our existence result treats the situation at the onset of facet breaking or right after it, in other words, the initial data are not general. In the simplest case we have three facets on each side of the bent rectangle. We recall that by a facet we understand a flat part of Γ whose normal vector is a singular direction of γ.

The variational solutions are 'semi-explicit' and this makes them easy to analyze. This also becomes a drawback in more complicated situations. This is exactly the reason why we consider, in [18], only a limited class of initial conditions. The difficulty is related to a variable character of the endpoints of the facets. This may be explained as follows. Once we write (1.1) in a local coordinate system, then we obtain a Hamilton–Jacobi equation with a free boundary—the facet endpoints. The free boundary is either a 'shock wave' or a 'rarefaction wave', depending upon the data. We have a strong feeling that growing complexity of cases that we study, calls for a new, more general tool.

Moreover, the uniqueness result in [18] is limited to a special configuration of the data. Here, we want to lift it, but only for bent rectangles, which are graphs of a piecewise C^1 functions over a smooth reference manifold. We present a geometric condition on the data which guarantees that such a manifold exists. This is done in Theorem 3.1 in Section 3. Once we reach that goal, we concentrate on showing that the variational solutions are indeed viscosity solutions in the sense of [16], developed for equations like

$$u_t = a(u_x)((W_p(u_x))_x + \sigma) \quad (x,t) \in \Omega \times (0,T) \tag{1.2}$$

augmented with periodic boundary conditions as well as initial data. In this equation W is a convex, continuous and piece-wise C^2 function.

However, we can only show that a profile function u (see Definition 4.3 for a rigorous statement) of a family $\{\Gamma(t)\}_{t\in[0,T)}$ of bent rectangles satisfies equation like (1.2), but the coefficient a depends not only on u_x but also on u and x in a non-trivial way, as well as W depends on x,

$$u_t = a(u_x,u,x)((W_p(u_x,x))_x + \sigma) \quad (x,t) \in \Omega \times (0,T), \tag{1.3}$$

see Theorem 3.2. This is not an obstacle for introducing the notion of viscosity solutions like in [16], but we have to check if the Comparison Principle, [16, Theorem 7] is still valid.

We show that the variational solutions to (1.1) constructed in [18] are viscosity solutions. This is done in Section 4. An easy part of this proof is done in [16, Section 5]. Here, we concentrate on the behavior of the vertex. In [18], we assumed that the vertices of $\Gamma(t)$ are defined as intersection of facets. Here, we look more closely at this issue pointing to the behavior of the kinetic coefficient, which is crucial to solve this issue.