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Asymptotic analysis of the non-steady Navier–Stokes equations in a tube structure. II. General case

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ABSTRACT

The non-steady Navier–Stokes equations with Dirichlet boundary conditions are considered in thin tube structures. These domains are connected finite unions of thin finite cylinders (in the $2D$ case respectively thin rectangles). The complete asymptotic expansion of the solution is constructed. It contains a regular part and three types of the boundary layer correctors: "in-space", "in-time" and "in-space-and-in-time". The estimates for the difference of the exact solution and its Jth asymptotic approximation are proved.

1. Introduction

The asymptotic behavior of the solutions to partial differential equations set in thin domains is intensively studied in the applied mathematics. Such domains may be rods, plates, tubes, periodic lattices, etc., as well as some unions of these elements (see [1–3,6–8,11–14]).

The studying of thin structures is motivated by the engineering of them as some industrial installations (frames, cranes) as well as by the biological applications [4,5,22], for example, the blood circulation system is a big union B_ε of thin cylindrical "vessels" and ε is the ratio of the radius and the height of the cylinders. This geometrical approximation for such flows was developed first in [13] for the steady Stokes and Navier–Stokes equations. However, the steady description is rather far from the real blood flow models. In particular, it is important to study periodic-in-time problems because of periodicity of the cardiac action. In the first part of the paper [18] we considered the case when the initial condition is homogeneous and the inflow/outflow $\mathbf{g}$ vanishes for small values of the time (i.e., as $0 \le t \le t^*;\ t^* > 0$). The results of [18] remain valid as well for the periodic-in-time case.

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In the present paper we remove this restriction, i.e., we consider the general case when the inflow/outflow function $\mathbf{g}$ does not vanish for small times ($\mathbf{g}$ is vanishing only for $t = 0$). In this case the complete asymptotic expansion of the solution contains four sums. The construction of the first two sums was described in the first part of the paper [18] and here is only shortly recalled. As it was described in the first part, with every edge of the graph of the structure we associate the asymptotic expansion from [15] (the first sum). This expansion is multiplied by a cut-off function vanishing near the nodes. This product is injected into the Navier–Stokes equations. The cut-off function generates a residual in the right-hand side "concentrated" near the nodes. We introduce then the boundary layer correctors (with respect to the space variables), in order to compensate this residual, and for these correctors we derive boundary value problems in infinite domains. These boundary layer-in-space functions form the second sum.

The approximation of the solution by these two sums is close to the exact solution in the case when $\mathbf{g}$ vanishes for small values of the time, but in the general case it gives a residual of order ε^2 for the initial condition. To remove this residual we construct below two other sums of the expansion: the boundary layer-in-time correctors for every edge (these terms are the analogues of the regular expansion) and the boundary layer-in-time-and-in-space correctors in the neighborhoods of all nodes and vertices (these terms are the analogues of the boundary-layer-in space expansion). All these new terms decay exponentially with respect to the fast time $\tau = t/\varepsilon^2$. As in [18] we get an equation on the graph for the macroscopic fast time pressure, but now this problem becomes non-steady and the theorems on the existence and uniqueness of the solution and its exponential decay with respect to the fast time become non-evident. This problem was studied in [17].

The overall residual after substitution of the exact solution by its asymptotic approximation of order J is calculated in Section 4.

Then arguing as in [18] we get the error estimates for the difference of the exact solution and its asymptotic approximation of order J.

In the Appendix we present results concerning the existence, uniqueness and asymptotic behavior of the solution to the non-steady Stokes problem in unbounded domains with cylindrical outlets to infinity. This problem appears as an auxiliary one in the procedure of construction of the boundary layers-in-time and the boundary layers-in-time-and-in-space.

As in [18] we will use the following notation. Let V be a Banach space. The norm of the element u in the function space V is denoted by $\|u\|_V$. Vector-valued functions are denoted by bold letters, and the spaces of scalar and vector-valued functions are not distinguished in notation. The vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \dots, n$, and $\|\mathbf{u}\|_V = \left(\sum_{i=1}^n \|u_i\|_V^2\right)^{1/2}$.

Let G be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, with the Lipschitz boundary ∂G. As usual, $C^\infty(G)$ is the set of all infinitely differentiable functions in G and $C_0^\infty(G)$ is the subset of functions from $C^\infty(G)$ with compact supports in G. We use the usual (see [10]) notation for the Sobolev and Hölder spaces: $W^{l,q}(G)$ and $C^{l+\delta}(\bar{G})$ ($l \geq 0$ is integer, $q \in [1, \infty)$, $\delta \in (0, 1)$). $\mathring{V}(G)$ always denotes the closure of $C_0^\infty(G)$ in the norm of V.

$L^q(0, T; V)$ is the space of functions u such that $u(\cdot, t) \in V$ for almost all $t \in [0, T]$ and the norm

$$\|u\|_{L^q(0,T;V)} = \left(\int_0^T \|u(\cdot, t)\|_V^q dt\right)^{\frac{1}{q}}$$

is finite.

2. Recalling the formulation of the problem

We consider the Navier–Stokes equations in a class of special domains called tube structures. These domains are connected finite unions of thin finite cylinders (in the $2D$ case respectively thin rectangles). Each such tube structure may be schematically represented by its graph: letting the thickness of tubes to tend to zero we find out that tubes degenerate to segments. Let us recall shortly the definition of these domains.